MAT 300: Test 1 – Solutions

1. *Let* $A = \{11, 22, 35\}$*,* $B = \{12, 11, 16\}$*,* $C = \{2, 35, 17, 22\}$*, and* $D = \{3, 8\}$*. Find the following sets.*

   (a) $A \cap B$
   $\{11\}$

   (b) $A - (B \cup C)$
   $\{11, 22, 35\} - \{11, 12, 16, 2, 35, 17, 22\} = \emptyset$

   (c) $A \times D$
   $\{(11, 3), (11, 8), (22, 3), (22, 8), (35, 3), (35, 8)\}$

   (d) $\wp(D)$ (i.e., the power set of D)
   $\{\emptyset, \{3\}, \{8\}, \{3, 8\}\}$

2. *Answers should be phrased as simply as possible.*

   (a) *State the negation of*

   For all $x \in \mathbb{Z}$*, we have* $x \in \mathbb{R}$ *and* $x^2 > 0$*.*

   Answer: There exists $x \in \mathbb{Z}$ such that $x \notin \mathbb{R}$ or $x^2 \leq 0$.

   (b) *State the negation of*

   For all $x \in \mathbb{R}$*, there exists* $y \in \mathbb{Z}$ *such that if* $xy \geq 15$*, then* $x < 11$ *or* $x = y^2$*.*

   Answer: There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{Z}$, $xy \geq 15$ and $x \geq 11$ and $x \neq y^2$.

   (c) *State the converse of*

   For all $x \in \mathbb{R}$*, if* $x^2 > 0$*, then* $x > 0$*.*

   Answer: For all $x \in \mathbb{R}$, if $x > 0$, then $x^2 > 0$.

3. *Make a complete truth table for the statements*

$$\text{“Not(P)} \implies \text{(P or Q)” and “(P and Q)} \iff \text{Q”.}$$

Then, in a sentence or two, explain if the two statements are logically equivalent, and refer to your table to justify your answer.

P	Q	Not P	P or Q	Not(P) $\implies$ (P or Q)	P and Q	(P and Q) $\iff$ Q
T	T	F	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	T	F	F
F	F	T	F	F	F	T

The statements are not logically equivalent because they have different truth values in the bottom two rows of the truth table as seen in columns 5 and 7.

4. *Prove that if $A \subseteq B$ and $C \subseteq D$, then $A \cap C \subseteq B \cap D$.*

   Let $x \in A \cap C$. Then $x \in A$ and $x \in B$. Since $x \in A$ and $A \subseteq B$, we have $x \in B$. Similarly, since $x \in C$ and $C \subseteq D$, we have $x \in D$. Thus, $x \in B$ and $x \in D \implies x \in B \cap D$.

   Therefore, $A \cap C \subseteq B \cap D$.

5. *Let A, B, and C be sets. Prove that if $A \neq \emptyset$ and $A \times B = A \times C$, then $B = C$.*

   Since $A \neq \emptyset$, there exists $a \in A$.

   Now, for all $b \in B$,

   $$\begin{aligned} (a,b) \in A \times B &\implies (a,b) \in A \times C \qquad\qquad \text{(because } A \times B = A \times C\text{)} \\ &\implies b \in C \end{aligned}$$

   Similarly, for all $c \in C$,

   $$\begin{aligned} (a,c) \in A \times C &\implies (a,c) \in A \times B \qquad\qquad \text{(because } A \times B = A \times C\text{)} \\ &\implies c \in B \end{aligned}$$

   Thus, $B = C$.

6. *Prove $\sqrt{14}$ is irrational.*

   Assume not. Then there exists integers a and b such that $\sqrt{14} = a/b$. Moreover, we can assume $b > 0$ and that b is the smallest such denominator.

   Then $14 = (a/b)^2 = a^2/b^2$. Multiplying by b^2 gives $14b^2 = a^2$. Since $14b^2 = 2(7b^2)$ is even, a^2 is even. By a proposition, a is even, and so $a = 2k$ for some integer k. Substituting, we get $14b^2 = (2k)^2 = 4k^2$. Dividing by 2 gives $7b^2 = 2k^2$, which is even. Since the product of two odd numbers is odd, and $7b^2$ is even (and 7 is odd), we must have b^2 even. Thus, $b = 2\ell$ for some integer ℓ. Moreover, $\sqrt{14} = a/b = (2k)/(2\ell) = k/\ell$.

   Now $\ell = b/2$ and $b > 0$, so $\ell > 0$ and $\ell < b$, contradicting the minimality of b. *

7. *Prove that for all $n \in \mathbb{Z}^+$,*

   $$\sum_{j=1}^{n} (6j - 1) = 3n^2 + 2n\,.$$

   We will use induction. Let $S(n)$ be the statement "$\sum_{j=1}^{n}(6j-1) = 3n^2 + 2n$".

   Basis step ($n = 1$):

   $$\sum_{j=1}^{1} (6j - 1) = 6 \cdot 1 - 1 = 5 = 3 \cdot 1^2 + 2 \cdot 1$$

Inductive step ($S(n-1) \implies S(n)$):

$$\begin{aligned}
\sum_{j=1}^{n}(6j-1) &= \left(\sum_{j=1}^{n-1}(6j-1)\right) + 6n - 1 \\
&= 3(n-1)^2 + 2(n-1) + (6n-1) && \text{(inductive hypothesis)} \\
&= 3(n^2 - 2n + 1) + 2n - 2 + 6n - 1 \\
&= 3n^2 + 2n
\end{aligned}$$