Homework 24

Let $[a, b] \subset \mathbf{R}$ be an interval, and $f \colon [a, b] \to \mathbf{R}$ a function. We say that f is *strictly concave* if for all $u, v \in [a, b]$,

$$\frac{f(u) + f(v)}{2} < f\left(\frac{u + v}{2}\right) .$$

(a) Prove that if f is continuous on $[a, b]$ and $f''(x) < 0$ for all $x \in (a, b)$, then f is strictly concave on $[a, b]$. (Hint: use the Mean Value Theorem a few times, starting with the intervals $[u, (u+v)/2]$ and $[(u+v)/2, v]$.)

(b) Prove that the function $f(x) = \log_2(x)$ is strictly concave on any interval $[a, b]$ where $0 < a < b$.

(c) Prove that the function

$$g(x) = \begin{cases} x \log_2(x) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous on $(0, 1)$.

(d) Prove that the function $g(x)$ from part (c) is strictly concave on $[0, 1]$.

Solutions for Parts b, c, and d

(b) Thanks to part (a), we just have to compute some derivatives. Recall that for any base m for a logarithm, $\log_m(x) = \frac{\ln(x)}{\ln(m)}$ and so $\frac{d}{dx}\log_m(x) = \frac{1}{x\ln(m)}$. So,

$$f'(x) = \frac{1}{x\ln(2)} = \frac{1}{\ln(2)}x^{-1} \quad \text{and} \quad f''(x) = \frac{-1}{\ln^2(2)}x^{-2} = -\left(\frac{1}{\ln(2)x}\right)^2$$

So, $f''(x) < 0$ for all $x \in \mathbf{R}^+$.

(c) Since $g(x)$ is differentiable on $(0, 2)$ (being the product of differentiable functions), it is continuous on $(0, 1]$. The only point we need to check is at 0. There,

$$\begin{aligned}
\lim_{x\to 0^+} g(x) &= \lim_{x\to 0^+} x\log_2(x) \\
&= \lim_{x\to 0^+} \frac{\log_2(x)}{1/x} \\
&= \lim_{x\to 0^+} \frac{1/(x\ln(2))}{-1/x^2} && \text{L'Hopital's Rule} \\
&= \lim_{x\to 0^+} -x/\ln(2) \\
&= 0 \\
&= g(0)
\end{aligned}$$

So, $g(x)$ is continuous at all points in $[0, 1]$.

(d) From part (c), $g(x)$ is continuous on $[0, 1]$. Then, on $(0, 1)$ we have more simply $g(x) = x\log_2(x)$, so

$$\begin{aligned}
g'(x) &= x\frac{1}{x\ln(2)} + \log_2(x) && \text{by the product rule} \\
&= \frac{1}{\ln(2)} + \log_2(x)
\end{aligned}$$

and

$$\begin{aligned} g''(x) &= 0 + \frac{-1}{x\ln(2)} \\ &= -\frac{1}{x\ln(2)} \end{aligned}$$

Note, $\ln(2) > 0$ (since $2 > 1$ and $\ln(x)$ is strictly increasing) and $x > 0$ since $x \in (0, 1)$, so $g''(x) < 0$. Thus, $g(x)$ is a strictly concave function.