Groups From Their Subgroup Diagrams

Understanding how constructions we have studied are reflected in the subgroup diagram of a group can reinforce the concepts. For example, we know that in a cyclic group of order n, there will be exactly one subgroup of order d for each $d \mid n$. Moreover, inclusions are given by divisibility of the orders of the subgroups.

In the first few cases where there is more than one group up to isomorphism, it is easy to distinguish groups. For example, V_4 has three subgroups of order 2 (as opposed to one for $\mathbf{Z}_4$) corresponding to the three elements of order 2. In order 6, S_3 has 3 subgroups of order 2 (as opposed to one for $\mathbf{Z}_6$) and they are conjugate. In order 8, the only group with non-normal subgroups is D_4, which is also distinguished by its 5 subgroups of order 2. The other 4 groups are distinguished by the number of subgroups of order 2 and 4. The case of order 10 is analogous to order 6: in both cases the groups can be taken to be D_n and $\mathbf{Z}_{2n}$ with n odd, and D_n then has n elements of order 2.

We can combine this with the correspondence between subgroups of a quotient group G/N and subgroups of G containing N. This correspondence preserves normality and conjugacy classes of subgroups.

1. In the groups of order 16, $\mathbf{Z}_4 : \mathbf{Z}_4$ has three normal subgroups of order 2. Based on the subgroup diagram of $\mathbf{Z}_4 : \mathbf{Z}_4$, identify the three corresponding quotient groups. For each one, explain what thing(s) in the subgroup diagram determined the quotient group.

2. For each of the three quotient groups identified in (1), click the button "Sub to Table" from the subgroup diagram to see the that quotient group in the group table. Verify that the quotient group you selected in (1) is correct based on the table.

3. In the groups of order 16, the group $\mathbf{Z}_4 \times \mathbf{Z}_2 : \mathbf{Z}_2$ has three normal subgroups of order 2. For each, identify the quotient group and explain what thing(s) in the subgroup diagram determined the quotient group.

4. In the groups of order 16, the group $\mathbf{Z}_4 \times \mathbf{Z}_2 : \mathbf{Z}_2$ has three normal subgroups of order 4. For each, identify the isomorphism type of both the subgroup and quotient group. In each case, explain what thing(s) in the subgroup diagram determined the subgroup and the quotient group.