Recognizing Direct Products

Recall that a group G is an internal direct product of H and K if H and K are subgroups of G and the map

$$\begin{aligned} f : H \times K &\to G \\ (h, k) &\mapsto hk \end{aligned}$$

is an isomorphism. One can often recognize if a group is an internal direct product by means of the following.

> If G is a group, G is the internal direct product of two subgroups H and K iff the following three conditions hold:
>
> (i) H and K are both normal subgroups,
>
> (ii) $H \cap K = \{e\}$,
>
> (iii) $HK = G$.

1. Prove that a group G is the internal direct product of two subgroups H and K iff the subgroups satisfy the three conditions above.

Given a group G, all of the conditions above are visible from a subgroup diagram for G provided the normal subgroups are marked since $H \cap K$ is the largest subgroup contained in both H and K, and since $K \trianglelefteq G$, HK is the smallest subgroup of G containing both H and K. Moreover, $HK \trianglelefteq G$ because both H and K are normal in G which may make it easier to find. So, one can focus on the normal subgroups of G containing both H and K.

To a certain extent, the conditions above are related to the orders of H, K, and G. This can simplify the task of looking for suitable H and K even more.

2. Suppose H is a subgroup of a finite group G and K is a normal subgroup of G such that $H \cap K = \{e\}$. Prove $HK = G$ iff $|K| = |G|/|H|$.

Now, look at subgroup diagrams (from the Group Table Calculator) to answer the following questions.

3. The group $\mathbf{Z}_4 \times \mathbf{Z}_2$ is clearly the direct product of two groups, but in how many ways can it be expressed as in internal direct product of H and K where $|H| = 4$ and $|K| = 2$?

4. Still with $\mathbf{Z}_4 \times \mathbf{Z}_2$, can any of the subgroups of order 4 be used for H? Can any of the subgroups of order 2 be used for K?

5. For each of the three non-abelian groups of order 12, which can be expressed as an internal direct product in a non-trivial way? (A trivial way would mean that either H or K is the trivial group.)

6. Express D_{10} as a non-trivial internal direct product. To what groups are H and K isomorphic?

7. Look at the subgroup diagram for $\mathbf{Z}_4 \times \mathbf{Z}_4$ and check to see that it cannot be expressed as the internal direct product of a subgroup of order 2 and a subgroup of order 8. (Note, this is guarenteed by the classification of finitely generated abelian groups, but it is still worth looking.)