Recognizing Semi-Direct Products

Recall that if H and N are groups and $\phi\colon H \to \mathrm{Aut}(N)$ is a homomorphism, then the semi-direct product of H and N by ϕ is a group, $N : H$ (or $N \rtimes H$) which has $N \times H$ as its underlying set and binary operation

$$(n_1, h_1) * (n_2, h_2) := (n_1 \phi(h_1)(n_2), h_1 h_2)$$

Here, $\phi(h_1) \in \mathrm{Aut}(N)$, so $\phi(h_1)(n_2) \in N$. Thus, the first coordinate on the right side is a product in N. This group has subgroups isomorphic to N and H given by $N' = \{(n, e_H) \mid n \in N\}$ and $H' = \{(e_N, h) \mid h \in H\}$ respectively.

1. Prove $N' \leq N : H$.
2. Prove $H' \leq N : H$.

There is a simple recognition theorem for semi-direct products:

> If G is a group, G is the semi-direct product of subgroups H and N iff the following three conditions hold:
>
> (i) N is a normal subgroup of G,
>
> (ii) $H \cap N = \{e\}$,
>
> (iii) $NH = G$.

These will be refered to below as the three conditions for a semi-direct product.

3. Prove that the subgroups H' and N' for $N : H$ satisfy the three conditions for a semi-direct product.

Note, these conditions are similar to those of a direct product.

4. Prove that if H and N satisfy the conditions above for semi-direct product, then G is an internal direct product of H and N iff H is also a normal subgroup of G.

Given a group G, all of the conditions above are visible from a subgroup diagram for G provided the normal subgroups are marked. As with recognizing direct products, $H \cap N$ is the largest subgroup contained in both H and N, and NH is the smallest subgroup containing both H and N (this using that N is a normal subgroup of G).

Again, as with direct products, if the conditions above hold, then $G \cong N : H$, so $|G| = |N : H| = |N \times H| = |N| \cdot |H|$. Thus, in considering a condidate for N, one should look for H where $|H| = |G|/|N|$. Moreover, if $H \cap N = \{e\}$ and $|G| = |N| \cdot |H|$, then $NH = G$.

Look at subgroup diagrams (from the Group Table Calculator) to answer the following questions.

5. Divide the groups of order less than 16 into the following three categories:

1. Is a non-trivial internal direct product of two subgroups.
2. Is not a non-trivial internal direct product of two subgroups, but is a non-trivial semi-direct product of two subgroups.
3. Is not a non-trivial semi-direct product of two subgroups.

Note, you should be able to use your knowledge of small groups to deal with some orders without consulting the Group Table Calculator.

6. Repeat exercise 5 for the groups of order 16.

7. Repeat exercise 5 for groups of order n with $17 \leq n \leq 21$.

8. Determine which dihedral groups are non-trivally semi-direct products, and then prove your answer.

9. Determine which of the groups $\mathrm{Aff}(\mathbf{Z}_n)$ are non-trivally semi-direct products, and then prove your answer.