Conjugacy Classes

Recall that if G is a group and $g_1, g_2 \in G$, we say that g_1 *is conjugate to* g_2 if there exists $a \in G$ such that $g_2 = a g_1 a^{-1}$. The set of elements conjugate to $g \in G$ then the *conjugacy class* of g, $\text{Conj}(g) = \{aga^{-1} \mid \text{for all } a \in G\}$.

1. Prove that "is conjugate to" is an equivalence relation on G, and that the conjugacy classes are the equivalence classes for this relation.
2. Prove that if G is abelian and $g \in G$, $\text{Conj}(g) = \{g\}$.
3. Prove that if G is a group, $g \in G$, and $h \in \text{Conj}(g)$, then $\text{Conj}(g) = \text{Conj}(h)$. (Hint: there is an easy proof using exercise 1.)

You can have the Group Calculator compute conjugacy classes by

(i) Select a group

(ii) Click on an element so that it is the only element highlighted

(iii) Click the "Close under Conj." button

Then the conjugacy class of the element you selected will be highlighted.

4. Look at the group A_4 (under groups of order 12). Verify exercise 3 for 3 pairs of elements.
5. For the group A_4, determine all of the conjugacy classes. (Note, thanks to exercise 3, the answer can be just a partition of elements: $\{i\}, \{(234), (134), \ldots\}$.)
6. Give the partition into conjugacy classes for D_4 and Q_8 in the groups of order 8.
7. Give the partition into conjugacy classes for $\mathit{Aff}(\mathbf{Z}_5)$.
8. Give the partition into conjugacy classes for the groups S_3 and S_4 (under groups of orders 6 and 24 respectively).
9. Based on the preceeding exercise, what might you conjecture for conjugacy classes in S_n in general? This is not much evidence, but the pattern is fairly simple.

The *centralizer* of an element $g \in G$ is

$$C_G(g) = \{a \in G \mid ag = ga\}$$

You can have the Group Calculator compute centralizers of elements by

(i) Select a group

(ii) Click on an element so that it is the only element highlighted

(iii) Click the "Centralizer" button

10. Prove that for any element g in a group G, $C_G(g)$ is a subgroup of G and $g \in C_G(g)$.

11. Prove that if $g \in G$, then $C_G(g) = G$ iff $g \in Z(G)$. (Here $Z(G) = \{g \in G \mid ga = ag \text{ for all } a \in G\}$ is the center of the group G.)

Conjugacy classes and centralizers are related by the following equation: for all $g \in G$,

$$[G : C_G(g)] = |\mathrm{Conj}(g)|. \tag{1}$$

The a general proof of equation (1) should appear in your text, probably applying a more general result on sizes of orbits for a group acting on a set.

12. Prove equation (1) directly in the case where $g \in Z(G)$.

13. Use equation (1) to prove that if G is finite and $g \in G$, then $|\mathrm{Conj}(g)|$ is a divisor of $|G|$. Compare this to sizes of conjugacy classes found in exercises 4–8.

We can see more examples equation (1) by consulting subgroup diagrams. In exercises 14–18, do the following.

(i) Find the size of the conjugacy class of the specified element $g \in G$ directly (using close under conjugation).

(ii) Find the cyclic subgroup $\langle g \rangle$ (select just g and then click "Generate Subgroup").

(iii) Click the "Subgroup to Diagram" button to switch to the subgroup diagram for G with the cyclic subgroup highlighted.

(iv) Try to identify $C_G(g)$. (From preceeding exercises, $g \in C_G(g)$ and $C_G(g) = G$ iff $g \in Z(G)$.)

(v) Compare your guess with a direct computation of $C_G(g)$ (done from the group table).

(vi) Verify that equation (1) holds.

14. $G = A_4$ and $g = (123)$

15. $G = D_5$ and $g = v_1$

16. $G = D_4$ and $g = |$

17. $G = \mathbf{Z}_8 :_3 \mathbf{Z}_2$ (in groups of order 16) and $g = (0; 1)$

18. $G = \mathbf{Z}_8 :_3 \mathbf{Z}_2$ and $g = (1; 0)$

19. Suppose n is odd, $R^j \in D_n$, and $1 \leq j < n$. Try values of n and j to see the size of $\text{Conj}(R^j)$ and make a conjecture. Then prove your conjecture (hint: determine $C_{D_n}(R^j)$).

20. Suppose n is odd, $v_j \in D_n$ a reflection, and $1 \leq j \leq n$. Try values of n and j to see the size of $\text{Conj}(v_j)$ and make a conjecture. Then prove your conjecture (hint: determine $C_{D_n}(v_j)$).

Finally, we make the connection to the class equation of a group.

21. Combine the fact that conjugacy classes partition elements of a group with equation (1) to deduce the following form of the *class equation* for a finite group G:

$$|G| = |Z(G)| + \sum_g [G : C_G(g)]$$

where the sum is over elements g representing distinct conjugacy classes of size larger than one.

22. Use the results of exercises 4–8 to verify the class equation for those groups.