Tables of elliptic curves over number fields

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Overview

1. Why make tables? What is a table?
2. Simple enumeration
3. Using modularity
4. Curves with prescribed primes of bad reduction, or known conductor or L-function
Why make tables of elliptic curves?

Since the early days of using computers in number theory, computations and tables have played an important part in experimentation, for the purpose of formulating, proving (and disproving) conjectures. This is particularly true in the study of elliptic curves.
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Early tables over number fields

What tables exist for elliptic curves over number fields (other than \(\mathbb{Q}\))?
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My PhD thesis (1981) contains 184 elliptic curves (in 138 isogeny classes) defined over five imaginary quadratic fields:

<table>
<thead>
<tr>
<th>Field</th>
<th>norm</th>
<th>#classes</th>
<th>#curves</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Q}(\sqrt{-1})$</td>
<td>500</td>
<td>$40 + 2 = 42$</td>
<td>$55 + 2 = 57$</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{-2})$</td>
<td>300</td>
<td>$36 + 4 = 40$</td>
<td>$51 + 4 = 55$</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{-3})$</td>
<td>500</td>
<td>$30 + 2 = 32$</td>
<td>$33 + 2 = 35$</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{-7})$</td>
<td>200</td>
<td>$18 + 1 = 19$</td>
<td>$30 + 1 = 31$</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{-11})$</td>
<td>200</td>
<td>$14 + 3 = 17$</td>
<td>$15 + 3 = 18$</td>
</tr>
</tbody>
</table>

These appeared in my first paper (Compositio 1984) with some additional (previously “missing”) curves added in 1987.
Why these fields and ranges?

For these five fields I had developed a theory of modular symbols and hence had computed, for levels $\mathfrak{n}$ whose norm is bounded as above, all rational cuspidal newforms of weight 2 for the Bianchi congruence subgroups $\Gamma_0(\mathfrak{n})$. 
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This explains the small norm bound! The computations were mostly done in 1980-81. The elliptic curves were mostly found by a simple search, and I could detect isogenies but not compute them, so these tables are not complete under isogeny.
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The combination of

1. modular form computations: which curves do we expect?
   and
2. targeted searching: which curves can we find?

will be an underlying theme of this talk.
Which curve models?

How do we specify an elliptic curve defined over a number field $K$? By a Weierstrass equation

$$Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6$$

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We still have to deal with scaling by units ($a_j \mapsto u^i a_j$), not a big issue for imaginary quadratic $K$, and by coordinate shifts we can assume that $a_1, a_2, a_3$ are reduced modulo $2, 3, 2$ (respectively).
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Over $\mathbb{Q}$ this gives unique “reduced minimal models”, but not over any other number fields except for the 7 imaginary quadratic fields $K$ with $h_K = 1$ and $\mathcal{O}_K^* = \{ \pm 1 \}$.
How do we order curves?

There are several possibilities:

- By height: say by $\max \{|a_1|, |a_2|, |a_3|, |a_4|, |a_6|\}$, or $\max \{|c_4|, |c_6|\}$, or $\max \{|c_{43}, |c_{62}|\}$: that is, by some suitably weighted height of the coefficient vector;
- By discriminant norm $N(\Delta_E)$;
- By conductor norm $N(n)$.

I will normally go for the last possibility, listing curves by conductor with conductors in order of norm. (Note that the number of isomorphism classes of curves with fixed conductor is finite.) Otherwise we might fix a set of primes $S$ and list all curves with good reduction outside $S$, again a finite set.
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Enumeration methods

This is the simplest approach:

- For each triple \((a_1, a_2, a_3) \in \mathcal{O}_K^3\) reduced modulo \((2, 3, 2)\) and for all \((a_4, a_6) \in \mathcal{O}_K^2\) with coefficients lying in a box (with respect to a fixed integral basis for \(\mathcal{O}_K\)), write down every equation!

- Throw away those not wanted (e.g. conductor norm too large) and keep the rest.

- Sort by conductor, isogeny class and isomorphism class.

- Compute curves isogenous to each found and kept.

This is slow! It is basically what I did in 1981 (with a small box!). Over \(\mathbb{Q}\) an efficient version of this was used to create the Stein-Watkins database.
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Modularity!

So we will come back to this after talking about modularity.
Modularity I: over $\mathbb{Q}$

Over $\mathbb{Q}$, while the first tables of elliptic curves were obtained by plain enumeration as described above, this was soon augmented by Birch’s student Tingley. In his 1975 thesis, Tingley used modular symbols to find elliptic curves of conductors up to about 300.
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Moreover: every elliptic curve over $\mathbb{Q}$ is modular [Wiles et al], i.e. is isogenous to one of the $E_f$. 
Modularity over $\mathbb{Q}$ (continued)

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So if we use modularity to compute elliptic curves over $\mathbb{Q}$, we will obtain complete tables for each conductor $N$.

This was not known when the tables were first published (Antwerp IV 1976 or AMEC 1st edn. 1992), but that did not stop the tables being welcomed, and useful.
Modularity over quadratic fields

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Computing the curves is still not as “easy” as over \( \mathbb{Q} \). . .
This brings us back to imaginary quadratic fields. Here, a lot less is known; and also, less is true!
Modularity over imaginary quadratic fields

Let $K$ be imaginary quadratic. For each “level” $n \triangleleft \mathcal{O}_K$ we define the congruence subgroup $\Gamma_0(n)$ of the Bianchi groups $\text{GL}_2(\mathcal{O}_K)$ in the normal way.
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I will omit all details of this today.

This has been implemented for a variety of imaginary quadratic fields by me and my students, and there is a version (based on some different ideas) by Dan Yasaki in Magma.

Today I will give you data for the first five fields only, for which I have most well-developed and efficient code (at https://github.com/JohnCremona/bianchi-progs).
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Imaginary quadratic newforms

In 2013 I extended my old (1981) tables of rational newforms over the first five imaginary quadratic fields to cover all levels of norm $\leq 10^4$ (and going further would not be hard):

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<tr>
<th>Field $\mathbb{Q}(\sqrt{-n})$</th>
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<th>#rational cuspidal newforms</th>
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<td>$\mathbb{Q}(\sqrt{-1})$</td>
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I also recruited a final year undergraduate, Warren Moore, who is doing his Masters dissertation on finding elliptic curves to match all these, or as many as he can, using whatever methods we can think of.

Here is how far he has got to (as of a few days ago):
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Imaginary quadratic progress

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Sieved enumeration using newform data

Each newform $F$ at level $n$ has an L-function $L(F, s)$ which “looks like” the (degree 4) L-function of an elliptic curve over $K$, with Euler product, analytic continuation to $\mathbb{C}$, functional equation. This is completely determined by knowing (1) the conductor $n$ and (2) the Hecke eigenvalues $a_p$ at primes $p$. 
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So in looking for the curves we can use a sieve as follows: take a small number $r$ of primes $p_i$ of degree 1, and do pre-computations so that it is very quick to compute the map

$$[a_1, a_2, a_3, a_4, a_6] \in \mathcal{O}_K^5 \mapsto \prod_i \mathbb{F}_p^5 \mapsto (a_{p_1}, \ldots, a_{p_r}) \in \mathbb{Z}^r$$

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Other tricks

- apply quadratic twists to existing curves
- work with $|a_{p_i}|$ to find twists with the sieve
- recognise base-change forms and use curves $/\mathbb{Q}$
- use Chinese remaindering to target hard-to-find curves in a larger box
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Over imaginary quadratic fields $K$, we do not expect an exact bijection between (a) rational cuspidal newforms of weight 2 for $\Gamma_0(n)$ over $K$ and (b) isogeny classes of elliptic curves $E$ of conductor $n$ defined over $K$! There are exceptions on both sides:
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For example the elliptic curve $Y^2 = X^3 - X$ (32a1) has $j = 1728$ and conductor $n = (64)$ over $\mathbb{Q}(\sqrt{-1})$, but $S_2(n)$ is trivial.
Exceptions II

- If $f \in S_2(N)$ over $\mathbb{Q}$ is quadratic and with “extra twist” by $K = \mathbb{Q}(\sqrt{-d})$: this means that the coefficients of $f$ lie in a (necessarily real) quadratic field $K_f = \mathbb{Q}(\sqrt{e})$ and

$$f \otimes \chi = f^{\sigma}$$

where $\chi$ is the quadratic character associated to $K/\mathbb{Q}$ and $\sigma$ generates $\text{Gal}(K_f/\mathbb{Q})$. Attached to $f$ is an abelian surface $A_f$ such that

$$\text{End}(A_f) \otimes \mathbb{Q} \cong \left(\frac{-d, e}{\mathbb{Q}}\right)$$

which may or may not split. If it does not then $A_f$ is absolutely simple. But the base-change of $f$ from $\mathbb{Q}$ to $K$ is a cusp form $F$ with rational coefficients, and this $F$ will then not have an associated elliptic curve.
Exceptions II example

In $S_2(3^5)$ (over $\mathbb{Q}$) there is a newform $f$ with

$$a_2 = \sqrt{6}, \quad a_5 = -\sqrt{6}, \quad a_7 = 2, \quad a_{11} = \sqrt{6};$$

in general, $a_p \in \mathbb{Z}$ for $p \equiv 1$ and $a_p/\sqrt{6} \in \mathbb{Z}$ for $p \equiv 2 \pmod{3}$. 

But the quaternion algebra $(-3, 6)\mathbb{Q}$ is not split so $A_f$ is simple (over $\mathbb{Q}(\sqrt{-3})$ and absolutely): there is no elliptic curve $E$ over $\mathbb{Q}(\sqrt{-3})$ with conductor $(81)$ such that $L(E, s) = L(F, s)$.

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This is the last method I will discuss today.
Curves with fixed conductor
(or set of primes of bad reduction)

If $S$ is a finite set of primes of the number field $K$ there are only finitely many (isomorphism classes of elliptic curves defined over $K$ with good reduction outside $S$.

Taking $S = \{p : p \mid n\}$ we can restrict to curves of conductor exactly $n$. 

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1. find all possible $j$-invariants;
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The second step is quite straightforward: assuming that $j \neq 0, 1728$ and that $\mathcal{S}$ contains all primes dividing 6, the curves form a complete set of quadratic twists by elements of $K(\mathcal{S}, 2)$.

For details see Cremona-Lingham 2007.
Finding the $j$-invariants for given $S$: I

[Continue to assume that $p \mid 6 \implies p \in S$ and $j \neq 0, 1728$.]

The $j$-invariants which occur are those such that

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If $w = j^2(j - 1728)^3 \in K(S, 6)_{12}$, take $u \in K^*$ such that $(3u)^6w \in K(S, 12)$; then

$$Y^2 = X^3 - 3u^2j(j - 1728)X - 2u^3j(j - 1728)^2$$

has good reduction outside $S$, and all such curves are twists of this by elements of $K(S, 2)$. 
Finding the $j$-invariants for given $S$: II

In Cremona-Lingham, one method for finding such $j$ is described: $j = x^3/w$ where $(x, y)$ is an $S$-integral point on $Y^2 = X^3 - 1728w$ with $w \in K(S, 6)_{12}$. 

I implemented this in Magma (and in Sage over $\mathbb{Q}$), but:

- The problem in practice is that finding all $S$-integral points on elliptic curves is not easy (and not implemented in general); my Magma code just searches for $S$-integral points.

+ The code has been used to find some of the "missing curves", over both real and imaginary quadratic fields (and has also been used for some higher degree fields).

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After I gave a talk about this at MSRI in 2010, Elkies had an idea: instead of finding the $j$-invariants directly, find the $\lambda$-invariants. This is now joint work with Angelos Koutsianas.

Recall that $j = 256\left(\lambda^2 - \lambda + 1\right)^3\lambda^2\left(\lambda - 1\right)^2$. This is the covering map from $X(2)$ to $X(1)$, both curves over $\mathbb{Q}$ of genus $0$.

If $j$ is an $S$-integer then $\lambda$ is an $S$-unit. And since $j(\lambda) = j(1 - \lambda)$ we also have that $1 - \lambda$ is an $S$-unit. So we can find $\lambda$, and hence $j$, by solving the $S$-unit equation $\lambda + \lambda = 1$.

There are now two sub-problems:

1. find all possible $2$-division fields $L = K(\lambda);
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The first problem has been solved, and implemented; we use Kummer Theory to find all extensions $L/K$ which are Galois, with group either $C_1$, $C_2$, $C_3$ or $S_3$. 

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