L-functions with rational integer coefficients

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joint work with
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Stefan Lemurell, Gothenburg University &
Chalmers University of Technology

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Curves and Automorphic Forms
Arizona State University
Wanted: table of hyperelliptic curves

a. ordered by conductor, \( N \)
b. certain none is missing

Available data:
M. Stoll

\[
\begin{align*}
169 & \quad y^2 = x^6 + 4x^5 + 6x^4 + 2x^3 + x^2 + 2x + 1 \\
249 & \quad y^2 = x^6 + 2x^3 + 4x^2 + 4x + 1 \\
277 & \quad y^2 = 4x^5 - 8x^3 + 8x^2 - 4x + 1 \\
295 & \quad y^2 = x^6 - 4x^4 - 2x^3 + 1
\end{align*}
\]

Stoll’s list does not attempt to be complete.
From Brumer and Kramer:

Table 2. Paramodular Abelian Surfaces of ODD Conductor $< 1000$.  

<table>
<thead>
<tr>
<th>$N$</th>
<th>EQUATION</th>
<th>INFO</th>
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<tbody>
<tr>
<td>249</td>
<td>$x^6 + 4x^5 + 4x^4 + 2x^3 + 1$</td>
<td>83</td>
</tr>
<tr>
<td>277</td>
<td>$x^6 + 2x^5 + 3x^4 + 4x^3 - x^2 - 2x + 1$</td>
<td>q</td>
</tr>
<tr>
<td>295</td>
<td>$x^6 - 2x^3 - 4x^2 + 1$</td>
<td>59</td>
</tr>
<tr>
<td>349</td>
<td>$x^6 - 2x^5 + 3x^4 - x^2 - 2x + 1$</td>
<td>q</td>
</tr>
<tr>
<td>353</td>
<td>$x^6 + 2x^5 + 5x^4 + 2x^3 + 2x^2 + 1$</td>
<td>wr72</td>
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<td>389</td>
<td>$x^6 + 2x^5 + 5x^4 + 8x^3 + 8x^2 + 4x$</td>
<td>389</td>
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<td>427</td>
<td>$x^6 - 4x^5 - 4x^4 + 18x^3 + 16x^2 - 16x - 15$</td>
<td>61</td>
</tr>
<tr>
<td>461</td>
<td>$x^6 + 2x^5 - 5x^4 - 8x^3 + 11x^2 + 10x - 11$</td>
<td>q</td>
</tr>
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<td>523</td>
<td>$x^6 - 2x^5 + x^4 + 4x^3 - 4x^2 - 4x$</td>
<td>523</td>
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<tr>
<td>555</td>
<td>$x^6 + 6x^5 + 5x^4 - 16x^3 - 8x^2 + 12x$</td>
<td>37</td>
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<tr>
<td>561</td>
<td><strong>PRYM</strong></td>
<td>11, 51</td>
</tr>
<tr>
<td>587a</td>
<td>$-3x^6 + 18x^4 + 6x^3 + 9x^2 - 54x + 57$</td>
<td>$S_6$, mild@3</td>
</tr>
<tr>
<td>587b</td>
<td>$x^6 + 2x^4 + 2x^3 - 3x^2 - 2x + 1$</td>
<td>$S_6$</td>
</tr>
<tr>
<td>597</td>
<td>$x^6 + 4x^5 + 8x^4 + 12x^3 + 8x^2 + 4x$</td>
<td>q</td>
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<tr>
<td>603</td>
<td>$x^6 - 4x^5 + 2x^4 + 4x^3 + x^2 - 4x$</td>
<td>67</td>
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<tr>
<td>623</td>
<td>$-224x^6 - 1504x^5 - 4448x^4 - 7200x^3 - 6080x^2 - 2048x$</td>
<td>89, mild@8</td>
</tr>
<tr>
<td>633</td>
<td>$24x^6 + 40x^5 + 28x^4 + 80x^3 + 52x^2 - 32x$</td>
<td>211, mild@2</td>
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<td>657</td>
<td><strong>WEIL RESTRICTION</strong></td>
<td>u, notSS</td>
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<tr>
<td>665</td>
<td><strong>PRYM</strong></td>
<td>19, 35</td>
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<td>691</td>
<td>$x^6 + 2x^5 - 3x^4 - 4x^3 + 4x$</td>
<td>691</td>
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<td>709</td>
<td>$-4x^5 - 7x^4 - 4x$</td>
<td>709</td>
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<tr>
<td>713a</td>
<td>$x^6 + 2x^5 + x^4 + 2x^3 - 2x^2 + 1$</td>
<td>23, 31</td>
</tr>
<tr>
<td>713b</td>
<td>$x^6 - 2x^5 + x^4 + 2x^3 + 2x^2 - 4x + 1$</td>
<td>23, 31</td>
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What is already known:

<table>
<thead>
<tr>
<th>Mestre</th>
<th>Stoll</th>
<th>Brumer &amp; Kramer</th>
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<tbody>
<tr>
<td>everything*</td>
<td>hyperelliptic curves</td>
<td>abelian surfaces</td>
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<tr>
<td>none with $N &lt; 107$</td>
<td>169</td>
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Goal: make a complete list up to $N_{\text{max}}$, say 300.

1. prove there are no other conductors, $N \leq N_{\text{max}}$
2. for each conductor found, find all the isomorphism classes
L-function axioms

- Dirichlet series
  \[ L(s) = \sum_{n=1}^{\infty} \frac{A_n/\sqrt{n}}{n^s} \quad A_n \in \mathbb{Z} \]

- Functional equation
  \[ \Lambda(s) := N^{s/2} \Gamma_C(s + \frac{1}{2})^2 L(s) = \pm \Lambda(1 - s) \]

- Euler product
  \[ L(s) = \prod_{p} F_p(p^{-s})^{-1}, \quad \text{where} \]
  \[ F_p(z) = G_p(z/\sqrt{p}) \quad \text{with} \quad G_p(z) \in \mathbb{Z}[z] \]
L-function axioms

- Dirichlet series \( L(s) = \sum_{n=1}^{\infty} \frac{A_n/\sqrt{n}}{n^s} \) \( A_n \in \mathbb{Z} \)

- Functional equation
  \[
  \Lambda(s) := N^{s/2} \Gamma_C(s + \frac{1}{2})^2 L(s) = \pm \Lambda(1 - s)
  \]

- Euler product
  \[
  L(s) = \prod_p F_p(p^{-s})^{-1}, \text{ where } F_p(z) = G_p(z/\sqrt{p}) \text{ with } G_p(z) \in \mathbb{Z}[z]
  \]

Furthermore

- \( F_p(0) = 1 \)

- If \( p \nmid N \) then \( F_p(z) \) has degree 4.
  If \( p \mid N \) then \( F_p(z) \) has degree \( \leq 3 \).

- If \( p \nmid N \) then all roots of \( F_p(z) \) lie on \( |z| = 1 \).

- If \( p \mid N \) then each root of \( F_p(z) \) lies on \( |z| = p^{m/2} \) for some \( m \in \{0, 1, 2, 3\} \).
Conclusion: there are only finitely many choices for $F_p$ for each $p$.

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<tr>
<th>prime, $p$</th>
<th>good</th>
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<tr>
<td>2</td>
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<td>7</td>
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The Method:
For a given $N$ and $\varepsilon = \pm 1$, everything about the L-function is known except the coefficients.

$\implies$ find coefficients such that the functional equation is true

OR

prove that the L-function cannot exist.
The Approximate Functional Equation

\( g(s): \) a nice test function

\[
\begin{align*}
\Lambda(s) &= \sum_{n=1}^{\infty} \frac{a_n}{n^s} \int_{(\nu)} f_{\text{compl.}}(g) + \sum_{n=1}^{\infty} \frac{\overline{a_n}}{n^{1-s}} \int_{(\nu)} f_{\text{compl.}}(g)
\end{align*}
\]

Solving for \( L(s) \) gives

\[
L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \int_{(\nu)} h_{\text{compl.}}(g) + \sum_{n=1}^{\infty} \frac{\overline{a_n}}{n^{1-s}} \int_{(\nu)} h_{\text{compl.}}(g)
\]

Choose \( s_0 \) and \( g_1(s) \):

\[
L(s_0) = \star + \star a_2 + \star a_3 + \star a_4 + \star a_5 + \star a_6 + \ldots
\]

Example: \( g(s) = e^{\frac{7}{8} s} \), if \( f \in S_{24}(\Gamma_0(1)) \) then

\[
L\left(\frac{1}{2}, f\right) = 1.473a_1 + 1.186a_2 - 0.0959a_3 - 0.00772a_4 + 0.000237a_5 + \ldots
\]
Our Method

\[ s_0, g_1 : L(s_0) = \star + \star a_2 + \star a_3 + \star a_4 + \star a_5 + \star a_6 + \ldots \]

\[ s_0, g_2 : L(s_0) = \star + \star a_2 + \star a_3 + \star a_4 + \star a_5 + \star a_6 + \ldots \]

Subtracting gives:

\[ 0 = \star + \star a_2 + \star a_3 + \star a_4 + \star a_5 + \star a_6 + \ldots \]

Apply Ramanujan bounds for \(a_j\) and check whether LHS = RHS.
Example: no hyperelliptic curve with \( N = 125 \) and \( \varepsilon = +1 \)

Choose \( s_0 = \frac{1}{2} \). Using the test function \( g(s) = 1 \):

\[
L\left(\frac{1}{2}\right) = 0.123 + 0.0241a_2 + 0.0058a_3 + 0.00195a_4 + 0.00075a_5 + \cdots
\]

and with \( g(s) = e^{\frac{1}{4}s} \):

\[
L\left(\frac{1}{2}\right) = 0.116 + 0.01834a_2 + 0.00444a_3 + 0.0013a_4 + 0.00044a_5 + \cdots
\]

Subtracting and rescaling:

\[
0 = 1 + 0.427a_2 + 0.187a_3 + 0.087a_4 + 0.043a_5 + 0.022a_6 + \cdots
\]

Try \( F_2(x) = 1 - 2\sqrt{2}x + 4x^2 - 2\sqrt{2}x^3 + x^4 \):

\[
0 = 2.57 + 0.254a_3 + 0.049a_5 + 0.0130a_7 + 0.00399a_9 + \cdots
\]

The Ramanujan bound is \( |a_p| \leq 4 \).

\[\implies\] Contradiction. \hspace{1cm} (34 more choices to try for \( F_2 \).)
Searching a tree:
Results:
For $N \leq 300$, $\varepsilon = +1$

eliminate all except:

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Results:
For $N \leq 300$, $\varepsilon = +1$

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$N = 121$ : $L(s, E_{11})$ $L(s, E_{11})$
Results:
For $N \leq 300$, $\varepsilon = +1$

eliminate all except:

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$N = 121$ : $L(s, E_{11})$ $L(s, E_{11})$
$N = 154$ : $L(s, E_{11})$ $L(s, E_{14})$
$N = 286$ : $L(s, E_{11})$ $L(s, E_{26})$ (two distinct L-functions)
Results:
For $N \leq 300$, $\varepsilon = +1$

eliminate all except:

121  154  165  169  187  196  209  210  220
225  231  238  249  255  256  264  266  277
280  285  286  289  294  295  297  300

product of two elliptic curve L-functions over $\mathbb{Q}$
Results:
For $N \leq 300$, $\varepsilon = +1$

eliminate all except:

121  154  165  169  187  196  209  210  220  225  231  238  249  255  256  264  266  277  280  285  286  289  294  295  297  300

product of two elliptic curve L-functions over $\mathbb{Q}$
hyperelliptic curve/product of conjugate cuspforms
hyperelliptic curve: \[ y^2 = x^6 + 4x^5 + 6x^4 + 2x^3 + x^2 + 2x + 1 \]

(Stoll)

But L-function is not primitive—it arises from two conjugate cuspforms of level 13:

\[ L(s) = L(s, f) \cdot L(s, \overline{f}). \]

Specifically,

\[ f_1, f_2 \in S_2(\Gamma_0(13), \chi_{13}(4, \cdot)) \]

2-dimensional

where \( \chi \) is the Dirichlet character defined by \( \chi_{13}(4, 2) = \frac{1}{2} + i\frac{\sqrt{3}}{2} \).
Now,

\[ f_1(q) = q + \left( \frac{-3 + i\sqrt{3}}{2} \right) q^2 + (-1 + i\sqrt{3}) q^3 + \cdots \]

\[ f_2(q) = q + \left( \frac{-3 - i\sqrt{3}}{2} \right) q^2 + (-1 - i\sqrt{3}) q^3 + \cdots \]

We find, in arithmetic normalisation,

\[ L(s) = 1 - \frac{3}{2^s} - \frac{2}{3^s} + \cdots \]

which agrees with the data from the LMFDB up to \( a_{19} \).
**Results:**

For $N \leq 300$, $\varepsilon = +1$

eliminate all except:

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product of two elliptic curve L-functions over $\mathbb{Q}$

hyperelliptic curve/product of conjugate cuspforms

hyperelliptic curve/abelian surface/paramodular form

*(Stoll; Brumer and Kramer; Poor and Yuen)*
Results:
For $N \leq 300$, $\varepsilon = +1$

eliminate all except:

121 154 165 169 187 196 209 210 220
225 231 238 249 255 256 264 266 277
280 285 286 289 294 295 297 300

product of two elliptic curve L-functions over $\mathbb{Q}$
hyperelliptic curve/product of conjugate cuspforms
hyperelliptic curve/abelian surface/paramodular form

(Stoll; Brumer and Kramer; Poor and Yuen)

What about $N = 256$?
$N = 256$

The L-function is not primitive; arises from two conjugate cuspforms of level 16.

$$L(s) = L(s, f) \ L(s, \overline{f})$$

Now,

$$f_1, f_2 \in S_2(\Gamma_0(16), \chi_{16}(11, \cdot)) \quad \text{2-dimensional}$$

where $\chi$ is the Dirichlet character defined by

$$\chi_{16}(11, 5) = i, \ \chi_{16}(11, 15) = 1$$

and $q$-expansion:

$$q + (-1 \mp i)q^2 + (-1 \pm i)q^3 \pm 2iq^4 + (-1 \mp i)q^5 + 2q^6 + \cdots.$$ 

Does it come from a hyperelliptic curve?
**Computational Theorem:** For $N \leq 300$ and $\varepsilon = +1$, the only levels for which there exist L-functions are those that were previously known.

$N = 407$ arises from $L(s, E_{11}) L(s, E_{37})$.

**Computational Theorem:** For $N \leq 450$ and $\varepsilon = -1$, the only level for which there exists an L-function is 407.
Final Questions

1. Does $N = 256$ come from a variety?

2. Why are there more non-primitive L-functions?

3. Why are small levels more likely to be odd?

4. Do all possible local factors eventually arise?
Stoll’s search method

\[ y^2 = x^6 + c_5 x^5 + c_4 x^4 + c_3 x^3 + c_2 x^2 + c_1 x + 1 \]

1. loop over curves where \( c_j \) are small (\( |c_j| \leq 5 \))
2. find the discriminant \( \Delta \); continue if not too big
3. determine the conductor, \( N \)
4. save the curve if \( N \) is small

Only \( N \) odd cases.

Curves with small \( N \) but large coefficients will be absent from list.