Modularity in Degree Two

Cris Poor
Fordham University

David S. Yuen
Lake Forest College

Curves and Automorphic Forms
Arizona State University, March 2014

What are the main ideas of this talk?

1. There is mounting evidence for the Paramodular Conjecture.
2. Borcherds products are a good way to make paramodular forms.

3. Our paramodular website exists: math.lfc.edu/∼yuen/paramodular

All elliptic curves $E/\mathbb{Q}$ are modular

Theorem (Wiles; Wiles and Taylor; Breuil, Conrad, Diamond and Taylor)

Let $N \in \mathbb{N}$. There is a bijection between

1. *isogeny classes of elliptic curves $E/\mathbb{Q}$ with conductor N*
2. *normalized Hecke eigenforms $f \in S_2(\Gamma_0(N))^{\mathrm{new}}$ with rational eigenvalues.*

In this correspondence we have $L(E, s, \mathrm{Hasse}) = L(f, s, \mathrm{Hecke})$.

All elliptic curves $E/\mathbb{Q}$ are modular

Theorem (Wiles; Wiles and Taylor; Breuil, Conrad, Diamond and Taylor)

Let $N \in \mathbb{N}$. There is a bijection between

1. *isogeny classes of elliptic curves $E/\mathbb{Q}$ with conductor N*
2. *normalized Hecke eigenforms $f \in S_2(\Gamma_0(N))^{\mathrm{new}}$ with rational eigenvalues.*

In this correspondence we have $L(E, s, \mathrm{Hasse}) = L(f, s, \mathrm{Hecke})$.

- Shimura proved 2 implies 1.

All elliptic curves $E/\mathbb{Q}$ are modular

Theorem (Wiles; Wiles and Taylor; Breuil, Conrad, Diamond and Taylor)

Let $N \in \mathbb{N}$. There is a bijection between

1. *isogeny classes of elliptic curves $E/\mathbb{Q}$ with conductor N*
2. *normalized Hecke eigenforms $f \in S_2(\Gamma_0(N))^{\mathrm{new}}$ with rational eigenvalues.*

In this correspondence we have $L(E, s, \mathrm{Hasse}) = L(f, s, \mathrm{Hecke})$.

- Shimura proved 2 implies 1.
- Weil added $N = N$.

All elliptic curves $E/\mathbb{Q}$ are modular

Theorem (Wiles; Wiles and Taylor; Breuil, Conrad, Diamond and Taylor)

Let $N \in \mathbb{N}$. There is a bijection between

1. *isogeny classes of elliptic curves $E/\mathbb{Q}$ with conductor N*
2. *normalized Hecke eigenforms $f \in S_2(\Gamma_0(N))^{\mathrm{new}}$ with rational eigenvalues.*

In this correspondence we have $L(E, s, \mathrm{Hasse}) = L(f, s, \mathrm{Hecke})$.

- Shimura proved 2 implies 1.
- Weil added $N = N$.
- Eichler (1954) proved the first examples $L(X_0(11), s, \mathrm{Hasse}) = L(\eta(\tau)^2\eta(11\tau)^2, s, \mathrm{Hecke})$.

All abelian surfaces $A/\mathbb{Q}$ are paramodular

Paramodular Conjecture (Brumer and Kramer 2009)

Let $N \in \mathbb{N}$. There is a bijection between

1. *isogeny classes of abelian surfaces $A/\mathbb{Q}$ with conductor N and endomorphisms $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$,*
2. *lines of Hecke eigenforms $f \in S_2(K(N))^{\mathrm{new}}$ that have rational eigenvalues and are not Gritsenko lifts from $J_{2,N}^{\mathrm{cusp}}$.*

In this correspondence we have

$$L(A, s, \text{Hasse-Weil}) = L(f, s, \text{spin}).$$

Remarks

- The paramodular group of level N,

$$K(N) = \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Q}), \qquad * \in \mathbb{Z},$$

Remarks

- The paramodular group of level N,

$$K(N) = \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Q}), \qquad * \in \mathbb{Z},$$

- $K(N)\backslash\mathcal{H}_2$ is a moduli space for complex abelian surfaces with polarization type $(1, N)$.

Remarks

- The paramodular group of level N,

$$K(N) = \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Q}), \qquad * \in \mathbb{Z},$$

- $K(N)\backslash\mathcal{H}_2$ is a moduli space for complex abelian surfaces with polarization type $(1, N)$.
- $K(N)$ is the stabilizer in $\mathrm{Sp}_2(\mathbb{Q})$ of $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus N\mathbb{Z}$.

Remarks

- The paramodular group of level N,

$$K(N) = \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Q}), \qquad * \in \mathbb{Z},$$

- $K(N)\backslash\mathcal{H}_2$ is a moduli space for complex abelian surfaces with polarization type $(1, N)$.
- $K(N)$ is the stabilizer in $\mathrm{Sp}_2(\mathbb{Q})$ of $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus N\mathbb{Z}$.
- New form theory for paramodular groups:
  Ibukiyama 1984; Roberts and Schmidt 2004, (LNM 1918).

Remarks

- The paramodular group of level N,

$$K(N) = \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Q}), \qquad * \in \mathbb{Z},$$

- $K(N)\backslash\mathcal{H}_2$ is a moduli space for complex abelian surfaces with polarization type $(1, N)$.
- $K(N)$ is the stabilizer in $\mathrm{Sp}_2(\mathbb{Q})$ of $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus N\mathbb{Z}$.
- New form theory for paramodular groups:
Ibukiyama 1984; Roberts and Schmidt 2004, (LNM 1918).
- Grit : $J_{k,N}^{\mathrm{cusp}} \to S_k\left(K(N)\right)$, the Gritsenko lift from Jacobi cusp forms of index N to paramodular cusp forms of level N is an advanced version of the Maass lift.

More Remarks

The subtle condition for general N: $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$.

- The endomorphisms that are defined over $\mathbb{Q}$ are trivial: $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$. This is the unknown case as well as the generic case in degree two. For elliptic curves it is always the case that $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$.

More Remarks

The subtle condition for general N: $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$.

- The endomorphisms that are defined over $\mathbb{Q}$ are trivial: $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$. This is the unknown case as well as the generic case in degree two. For elliptic curves it is always the case that $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$.
- Yoshida 1980 conjectured *All abelian surfaces $A/\mathbb{Q}$ are modular for weight two and some discrete subgroup*, and gave examples for $\Gamma_0^{(2)}(p)$ where A has conductor p^2 and $\mathrm{End}_{\mathbb{Q}}(A)$ is an order in a quadratic field and the Siegel modular form is a *Yoshida* lift.

More Remarks

The subtle condition for general N: $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$.

- The endomorphisms that are defined over $\mathbb{Q}$ are trivial: $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$. This is the unknown case as well as the generic case in degree two. For elliptic curves it is always the case that $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$.
- Yoshida 1980 conjectured *All abelian surfaces $A/\mathbb{Q}$ are modular for weight two and some discrete subgroup*, and gave examples for $\Gamma_0^{(2)}(p)$ where A has conductor p^2 and $\mathrm{End}_{\mathbb{Q}}(A)$ is an order in a quadratic field and the Siegel modular form is a *Yoshida* lift.
- Give credit to Brumer. Prior to the Paramodular Conjecture, I would have guessed that modularity in degree two would mainly involve the groups $\Gamma_0^{(2)}(N)$.

All abelian surfaces $A/\mathbb{Q}$ are paramodular

Maybe you want to see the Paramodular Conjecture again after the remarks

Paramodular Conjecture

Let $N \in \mathbb{N}$. There is a bijection between

1. *isogeny classes of abelian surfaces $A/\mathbb{Q}$ with conductor N and endomorphisms $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$,*
2. *lines of Hecke eigenforms $f \in S_2(K(N))^{\mathrm{new}}$ that have rational eigenvalues and are not Gritsenko lifts from $J_{2,N}^{\mathrm{cusp}}$.*

In this correspondence we have

$$L(A, s, \text{Hasse-Weil}) = L(f, s, \text{spin}).$$

Do the arithmetic and automorphic data match up?

Looks like it.

1997: Brumer makes a (short) list of $N < 1,000$ that could possibly be the conductor of an abelian surface $A/\mathbb{Q}$.

Theorem (PY 2009)

Let $p < 600$ be prime. If $p \notin \{277, 349, 353, 389, 461, 523, 587\}$ then $S_2(K(p))$ consists entirely of Gritsenko lifts.

This exactly matches Brumer's "Yes" list for prime levels.

This is a lot of evidence for the Paramodular Conjecture because prime levels $p < 600$ that don't have abelian surfaces over $\mathbb{Q}$ also don't have any paramodular cusp forms beyond the Gritsenko lifts.

Proof.

We can inject the weight two space into weight four spaces:

1) For $g_1, g_2 \in \text{Grit}\left(J_{2,p}^{\text{cusp}}\right) \subseteq S_2\left(K(p)\right)$, we have the injection:

$$\begin{aligned} S_2(K(p)) &\hookrightarrow \{(H_1, H_2) \in S_4(K(p)) \times S_4(K(p)) : g_2 H_1 = g_1 H_2\} \\ f &\mapsto (g_1 f, g_2 f) \end{aligned}$$

2) The dimensions of $S_4(K(p))$ are known by Ibukiyama; we still have to span $S_4(K(p))$ by computing products of Gritsenko lifts, traces of theta series and by smearing with Hecke operators.

3) Millions of Fourier coefficients mod 109 later,

$$\dim S_2(K(p)) \leq \dim\{(H_1, H_2) \in S_4(K(p)) \times S_4(K(p)) : g_2 H_1 = g_1 H_2\}$$

□

Examples of nonlifts are naturally more interesting

Method of Integral Closure

Theorem (PY 2009)

We have $\dim S_2(K(277)) = 11$ *but* $\dim J_{2,277}^{\mathrm{cusp}} = 10$. *There is a Hecke eigenform* $f_{277} \in S_2(K(277))$ *that is not a Gritsenko lift.*

- $\mathcal{A}_{277}$ is the Jacobian of the hyperelliptic curve

$$y^2 + y = x^5 + 5x^4 + 8x^3 + 6x^2 + 2x$$

Examples of nonlifts are naturally more interesting

Method of Integral Closure

Theorem (PY 2009)

We have $\dim S_2(K(277)) = 11$ *but* $\dim J_{2,277}^{\text{cusp}} = 10$. *There is a Hecke eigenform* $f_{277} \in S_2(K(277))$ *that is not a Gritsenko lift.*

- $\mathcal{A}_{277}$ is the Jacobian of the hyperelliptic curve

$$y^2 + y = x^5 + 5x^4 + 8x^3 + 6x^2 + 2x$$

- Magma will compute lots of Euler factors for $L(\mathcal{A}_{277}, s, \text{H-W})$

Examples of nonlifts are naturally more interesting

Method of Integral Closure

Theorem (PY 2009)

We have $\dim S_2(K(277)) = 11$ *but* $\dim J_{2,277}^{\text{cusp}} = 10$. *There is a Hecke eigenform* $f_{277} \in S_2(K(277))$ *that is not a Gritsenko lift.*

- $\mathcal{A}_{277}$ is the Jacobian of the hyperelliptic curve

$$y^2 + y = x^5 + 5x^4 + 8x^3 + 6x^2 + 2x$$

- Magma will compute lots of Euler factors for $L(\mathcal{A}_{277}, s, \text{H-W})$
- By contrast, we can only compute three Euler factors of $L(f_{277}, s, \text{spin})$.

Examples of nonlifts are naturally more interesting

Method of Integral Closure

Theorem (PY 2009)

We have $\dim S_2(K(277)) = 11$ *but* $\dim J_{2,277}^{\mathrm{cusp}} = 10$. *There is a Hecke eigenform* $f_{277} \in S_2(K(277))$ *that is not a Gritsenko lift.*

- $\mathcal{A}_{277}$ is the Jacobian of the hyperelliptic curve

$$y^2 + y = x^5 + 5x^4 + 8x^3 + 6x^2 + 2x$$

- Magma will compute lots of Euler factors for $L(\mathcal{A}_{277}, s, \mathrm{H\text{-}W})$
- By contrast, we can only compute three Euler factors of $L(f_{277}, s, \mathrm{spin})$.
- But they agree! The 2, 3 and 5 Euler factors of $L(f_{277}, s, \mathrm{spin})$ agree with those of $L(\mathcal{A}_{277}, s, \mathrm{H\text{-}W})$.

Examples of nonlifts are naturally more interesting

Method of Integral Closure

Theorem (PY 2009)

We have $\dim S_2(K(277)) = 11$ *but* $\dim J^{\mathrm{cusp}}_{2,277} = 10$. *There is a Hecke eigenform* $f_{277} \in S_2(K(277))$ *that is not a Gritsenko lift.*

- $\mathcal{A}_{277}$ is the Jacobian of the hyperelliptic curve

$$y^2 + y = x^5 + 5x^4 + 8x^3 + 6x^2 + 2x$$

- Magma will compute lots of Euler factors for $L(\mathcal{A}_{277}, s, \mathrm{H\text{-}W})$
- By contrast, we can only compute three Euler factors of $L(f_{277}, s, \mathrm{spin})$.
- But they agree! The 2, 3 and 5 Euler factors of $L(f_{277}, s, \mathrm{spin})$ agree with those of $L(\mathcal{A}_{277}, s, \mathrm{H\text{-}W})$.
- Do you want to *see* f_{277}? Later, when we have *theta blocks*.

How can we prove a weight two nonlift cusp form exists?

Method of Integral Closure

Proof.

1) We have a candidate $f = H_1/g_1 \in M_2^{\mathrm{mero}}(K(p))$.

2) Find a weight four cusp form $F \in S_4(K(p))$ and prove

$$F\, g_1^2 = H_1^2 \text{ in } S_8(K(p)).$$

Since $F = \left(\frac{H_1}{g_1}\right)^2$ is holomorphic, so is $f = \frac{H_1}{g_1}$.

□

How can we prove a weight two nonlift cusp form exists?

Method of Integral Closure

Proof.

1) We have a candidate $f = H_1/g_1 \in M_2^{\mathrm{mero}}(K(p))$.

2) Find a weight four cusp form $F \in S_4(K(p))$ and prove

$$F\, g_1^2 = H_1^2 \text{ in } S_8(K(p)).$$

Since $F = \left(\frac{H_1}{g_1}\right)^2$ is holomorphic, so is $f = \frac{H_1}{g_1}$. □

The *GROAN* you hear is the computer chugging away in weight 8.

Those whose strength gives out fall down along the way.

Confucius, *The Analects*

- What about $349^+, 353^+, 389^+, 461^+, 523^+, 587^+, 587^-$?
- The method of integral closure has only been used to prove existence of a nonlift for $f_{277} \in S_2(K(277))^+$ where $\dim S_8(K(277)) = 2529$.
- Spanning more weight eight spaces was too expensive for us.
- We told our troubles to V. Gritsenko and he suggested 587^- might give a Borcherds Products. And that is what the rest of this talk is about.

Those whose strength gives out fall down along the way.

Confucius, *The Analects*

- What about $349^+, 353^+, 389^+, 461^+, 523^+, 587^+, 587^-$?

- The method of integral closure has only been used to prove existence of a nonlift for $f_{277} \in S_2(K(277))^+$ where $\dim S_8(K(277)) = 2529$.

- Spanning more weight eight spaces was too expensive for us.

- We told our troubles to V. Gritsenko and he suggested 587^- might give a Borcherds Products. And that is what the rest of this talk is about.

But first, report on recent evidence from other sources.

Central L-values

Paramodular Boecherer Conjecture (Ryan and Tornaria 2011)

Let p be prime and k be even. Let $f \in S_k(K(p))$ be a cuspidal Hecke eigenform with Fourier expansion

$$f(Z) = \sum_{T>0} a(T;f)e(\mathrm{tr}(ZT)).$$

There exists a constant c_f such that for every fund. disc. $D < 0$,

$$\rho_o L(f, \tfrac{1}{2}, \chi_D)|D|^{k-1} = c_f \left(\sum_{[T] \text{ disc. } D} \frac{1}{\epsilon(T)} a(T;f) \right)^2,$$

where $\epsilon(T) = |\mathrm{Aut}_{\Gamma_0(p)}(T)|$ and $\rho_o = 1$ or 2 as $(p, D) = 1$ or $p|D$.

Central L-values

Paramodular Boecherer Conjecture (Ryan and Tornaria 2011)

Let p be prime and k be even. Let $f \in S_k(K(p))$ be a cuspidal Hecke eigenform with Fourier expansion

$$f(Z) = \sum_{T>0} a(T;f)e(\mathrm{tr}(ZT)).$$

There exists a constant c_f such that for every fund. disc. $D < 0$,

$$\rho_o L(f, \tfrac{1}{2}, \chi_D)|D|^{k-1} = c_f \left(\sum_{[T] \text{ disc. } D} \frac{1}{\epsilon(T)} a(T;f) \right)^2,$$

where $\epsilon(T) = |\mathrm{Aut}_{\Gamma_0(p)}(T)|$ and $\rho_o = 1$ or 2 as $(p, D) = 1$ or $p|D$.

- Proven for Gritsenko lifts.

Central L-values

Paramodular Boecherer Conjecture (Ryan and Tornaria 2011)

Let p be prime and k be even. Let $f \in S_k(K(p))$ be a cuspidal Hecke eigenform with Fourier expansion

$$f(Z) = \sum_{T>0} a(T;f)e(\mathrm{tr}(ZT)).$$

There exists a constant c_f such that for every fund. disc. $D < 0$,

$$\rho_o L(f, \tfrac{1}{2}, \chi_D)|D|^{k-1} = c_f \left(\sum_{[T] \text{ disc. } D} \frac{1}{\epsilon(T)} a(T;f) \right)^2,$$

where $\epsilon(T) = |\operatorname{Aut}_{\Gamma_0(p)}(T)|$ and $\rho_o = 1$ or 2 as $(p, D) = 1$ or $p|D$.

- Proven for Gritsenko lifts.
- Tested using Brumer's curves and our Fourier coefficients.

Equality of L-series

Complete Examples

Theorem Report (Johnson-Leung and Roberts 2012)

Let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field. Given a weight (k, k) Hilbert modular form h, with a linearly independent conjugate, they figured out how to lift h to a paramodular Hecke eigenform of level $\mathrm{Norm}(\mathbf{n})d^2$ with corresponding eigenvalues.

- Let E/K be an elliptic curve not isogenous to its conjugate.
- Let $A/\mathbb{Q}$ be the abelian surface given by the Weil restriction of E. Defining property: $A(\mathbb{Q})$ corresponds to $E(K)$
- Assume we know that E/K is modular w.r.t. a Hilbert form h.
- Then $A/\mathbb{Q}$ is modular w.r.t. the Johnson-Leung Roberts lift of h.
- Dembélé and Kumar have a preprint about this.

Equality of L-series

Complete Examples

Theorem Report (Johnson-Leung and Roberts 2012)

Let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field. Given a weight (k, k) Hilbert modular form h, with a linearly independent conjugate, they figured out how to lift h to a paramodular Hecke eigenform of level $\mathrm{Norm}(\mathbf{n})d^2$ with corresponding eigenvalues.

- Let E/K be an elliptic curve not isogenous to its conjugate.
- Let $A/\mathbb{Q}$ be the abelian surface given by the Weil restriction of E. Defining property: $A(\mathbb{Q})$ corresponds to $E(K)$
- Assume we know that E/K is modular w.r.t. a Hilbert form h.
- Then $A/\mathbb{Q}$ is modular w.r.t. the Johnson-Leung Roberts lift of h.
- Dembélé and Kumar have a preprint about this.
- For a similar but different example: Berger, Dembélé, Pacetti, Sengun for $N = 223^2$ and K imaginary quadratic.

Definition of Siegel Modular Form

- Siegel Upper Half Space: $\mathcal{H}_n = \{Z \in M^{\mathrm{sym}}_{n\times n}(\mathbb{C}) : \operatorname{Im} Z > 0\}$.
- Symplectic group: $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{R})$ acts on $Z \in \mathcal{H}_n$ by $\sigma \cdot Z = (AZ + B)(CZ + D)^{-1}$.
- $\Gamma \subseteq \mathrm{Sp}_n(\mathbb{R})$ such that $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ has finite index in Γ and $\mathrm{Sp}_n(\mathbb{Z})$
- Siegel Modular Form: $M_k(\Gamma) = \{$ holomorphic $f : \mathcal{H}_n \to \mathbb{C}$ that transforms by $\det(CZ + D)^k$ and are "bounded at the cusps" $\}$
- Cusp Form: $S_k(\Gamma) = \{f \in M_k(\Gamma)$ that "vanish at the cusps"$\}$
- Fourier Expansion: $f(Z) = \sum_{T \geq 0} a(T; f) e(\mathrm{tr}(ZT))$
- $n = 2$; $\Gamma = K(N)$; $T \in \begin{pmatrix} \mathbb{Z} & \frac{1}{2}\mathbb{Z} \\ \frac{1}{2}\mathbb{Z} & N\mathbb{Z} \end{pmatrix}$

Examples of Siegel Modular Forms

- Thetanullwerte: $\theta \begin{bmatrix} a \\ b \end{bmatrix} (0, Z) \in M_{1/2}\left(\Gamma^{(n)}(8)\right)$ for $a, b \in \frac{1}{2}\mathbb{Z}^n$
- Riemann Theta Function:

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, Z) = \sum_{m \in \mathbb{Z}^n} e\left(\tfrac{1}{2}(m+a)' Z (m+a) + (m+a)'(z+b)\right)$$

- $X_{10} = \prod_{a,b}^{10} \theta \begin{bmatrix} a \\ b \end{bmatrix} (0, Z)^2 \in S_{10}(\mathrm{Sp}_2(\mathbb{Z})) \quad (4a \cdot b \equiv 0 \mod 4)$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1/2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix},$$
$$\begin{bmatrix} 1/2 & 1/2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1/2 & 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}, \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

Definition of Jacobi Forms: Automorphicity

Level one

- Assume $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ is holomorphic.

$$\tilde{\phi} : \mathcal{H}_2 \to \mathbb{C}$$
$$\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \mapsto \phi(\tau, z) e(m\omega)$$

- Assume that $\tilde{\phi}$ transforms by $\chi \det(CZ + D)^k$ for

$$P_{2,1}(\mathbb{Z}) = \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Z}), \qquad * \in \mathbb{Z},$$

Definition of Jacobi Forms: Automorphicity

Level one

- Assume $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ is holomorphic.

$$\tilde{\phi} : \mathcal{H}_2 \to \mathbb{C}$$
$$\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \mapsto \phi(\tau, z) e(m\omega)$$

- Assume that $\tilde{\phi}$ transforms by $\chi \det(CZ + D)^k$ for

$$P_{2,1}(\mathbb{Z}) = \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Z}), \qquad * \in \mathbb{Z},$$

- $P_{2,1}(\mathbb{Z})/\{\pm I\} \cong \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathrm{Heisenberg}(\mathbb{Z})$

Definition of Jacobi Forms: Support

- Jacobi forms are tagged with additional adjectives to reflect the support $\mathrm{supp}(\phi) = \{(n, r) \in \mathbb{Q}^2 : c(n, r; \phi) \neq 0\}$ of the Fourier expansion

$$\phi(\tau, z) = \sum_{n,r \in \mathbb{Q}} c(n, r; \phi) q^n \zeta^r, \qquad q = e(\tau), \zeta = e(z).$$

- $\phi \in J_{k,m}^{\mathrm{cusp}}$: automorphicity and $c(n, r; \phi) \neq 0 \implies 4mn - r^2 > 0$

Definition of Jacobi Forms: Support

- Jacobi forms are tagged with additional adjectives to reflect the support $\operatorname{supp}(\phi) = \{(n,r) \in \mathbb{Q}^2 : c(n,r;\phi) \neq 0\}$ of the Fourier expansion

$$\phi(\tau, z) = \sum_{n,r \in \mathbb{Q}} c(n,r;\phi) q^n \zeta^r, \qquad q = e(\tau), \zeta = e(z).$$

- $\phi \in J_{k,m}^{\text{cusp}}$: automorphicity and $c(n,r;\phi) \neq 0 \implies 4mn - r^2 > 0$
- $\phi \in J_{k,m}$: automorphicity and $c(n,r;\phi) \neq 0 \implies 4mn - r^2 \geq 0$

Definition of Jacobi Forms: Support

- Jacobi forms are tagged with additional adjectives to reflect the support $\mathrm{supp}(\phi) = \{(n,r) \in \mathbb{Q}^2 : c(n,r;\phi) \neq 0\}$ of the Fourier expansion

$$\phi(\tau,z) = \sum_{n,r\in\mathbb{Q}} c(n,r;\phi) q^n \zeta^r, \qquad q = e(\tau), \zeta = e(z).$$

- $\phi \in J^{\mathrm{cusp}}_{k,m}$: automorphicity and $c(n,r;\phi) \neq 0 \implies 4mn - r^2 > 0$
- $\phi \in J_{k,m}$: automorphicity and $c(n,r;\phi) \neq 0 \implies 4mn - r^2 \geq 0$
- $\phi \in J^{\mathrm{weak}}_{k,m}$: automorphicity and $c(n,r;\phi) \neq 0 \implies n \geq 0$

Definition of Jacobi Forms: Support

- Jacobi forms are tagged with additional adjectives to reflect the support $\mathrm{supp}(\phi) = \{(n,r) \in \mathbb{Q}^2 : c(n,r;\phi) \neq 0\}$ of the Fourier expansion

$$\phi(\tau, z) = \sum_{n,r \in \mathbb{Q}} c(n,r;\phi) q^n \zeta^r, \qquad q = e(\tau), \zeta = e(z).$$

- $\phi \in J^{\mathrm{cusp}}_{k,m}$: automorphicity and $c(n,r;\phi) \neq 0 \implies 4mn - r^2 > 0$
- $\phi \in J_{k,m}$: automorphicity and $c(n,r;\phi) \neq 0 \implies 4mn - r^2 \geq 0$
- $\phi \in J^{\mathrm{weak}}_{k,m}$: automorphicity and $c(n,r;\phi) \neq 0 \implies n \geq 0$
- $\phi \in J^{\mathrm{wh}}_{k,m}$: automorphicity and $c(n,r;\phi) \neq 0 \implies n >> -\infty$ ("wh" stands for *weakly holomorphic*)

Examples of Jacobi Forms

- Dedekind Eta function $\eta \in J^{\text{cusp}}_{1/2,0}(\epsilon)$

$$\eta(\tau) = \sum_{n\in\mathbb{Z}} \left(\frac{12}{n}\right) q^{n^2/24} = q^{1/24} \prod_{n\in\mathbb{N}} (1-q^n)$$

- Odd Jacobi Theta function $\vartheta \in J^{\text{cusp}}_{1/2,1/2}(\epsilon^3 v_H)$

$$\vartheta(\tau,z) = \sum_{n\in\mathbb{Z}} \left(\frac{-4}{n}\right) q^{n^2/8}\zeta^{n/2}$$
$$= q^{1/8}\left(\zeta^{1/2}-\zeta^{-1/2}\right)\prod_{n\in\mathbb{N}}(1-q^n)(1-q^n\zeta)(1-q^n\zeta^{-1})$$

Examples of Jacobi Forms

- Dedekind Eta function $\eta \in J^{\text{cusp}}_{1/2,0}(\epsilon)$

$$\eta(\tau) = \sum_{n\in\mathbb{Z}} \left(\frac{12}{n}\right) q^{n^2/24} = q^{1/24} \prod_{n\in\mathbb{N}} (1-q^n)$$

- Odd Jacobi Theta function $\vartheta \in J^{\text{cusp}}_{1/2,1/2}(\epsilon^3 v_H)$

$$\vartheta(\tau,z) = \sum_{n\in\mathbb{Z}} \left(\frac{-4}{n}\right) q^{n^2/8}\zeta^{n/2}$$
$$= q^{1/8}\left(\zeta^{1/2} - \zeta^{-1/2}\right) \prod_{n\in\mathbb{N}} (1-q^n)(1-q^n\zeta)(1-q^n\zeta^{-1})$$

- $\vartheta_\ell \in J^{\text{cusp}}_{1/2,\ell^2/2}(\epsilon^3 v_H^\ell)$, $\quad \vartheta_\ell(\tau,z) = \vartheta(\tau,\ell z)$

Theta Blocks

A theory due to Gritsenko, Skoruppa and Zagier.

Definition

A theta block is a function $\eta^{c(0)} \prod_\ell \left(\frac{\vartheta_\ell}{\eta}\right)^{c(\ell)} \in J_{k,m}^{\mathrm{mero}}$ for a sequence $c : \mathbb{N} \cup \{0\} \to \mathbb{Z}$ with finite support.

- There is a famous Jacobi form of weight two and index 37:

$$f_{37} = \frac{\vartheta_1^3 \vartheta_2^3 \vartheta_3^2 \vartheta_4 \vartheta_5}{\eta^6} = \mathrm{TB}_2[1, 1, 1, 2, 2, 2, 3, 3, 4, 5].$$

Theta Blocks

A theory due to Gritsenko, Skoruppa and Zagier.

Definition

A theta block is a function $\eta^{c(0)} \prod_\ell \left(\frac{\vartheta_\ell}{\eta}\right)^{c(\ell)} \in J_{k,m}^{\mathrm{mero}}$ for a sequence $c : \mathbb{N} \cup \{0\} \to \mathbb{Z}$ with finite support.

- There is a famous Jacobi form of weight two and index 37:

$$f_{37} = \frac{\vartheta_1^3 \vartheta_2^3 \vartheta_3^2 \vartheta_4 \vartheta_5}{\eta^6} = \mathrm{TB}_2[1,1,1,2,2,2,3,3,4,5].$$

- $\prod_{\ell \in [1,1,1,2,2,2,3,3,4,5]} \left(\zeta^{\ell/2} - \zeta^{-\ell/2}\right)$, the *baby* theta block.

Theta Blocks

A theory due to Gritsenko, Skoruppa and Zagier.

Definition

A theta block is a function $\eta^{c(0)} \prod_\ell \left(\frac{\vartheta_\ell}{\eta}\right)^{c(\ell)} \in J_{k,m}^{\mathrm{mero}}$ for a sequence $c : \mathbb{N} \cup \{0\} \to \mathbb{Z}$ with finite support.

- There is a famous Jacobi form of weight two and index 37:

$$f_{37} = \frac{\vartheta_1^3 \vartheta_2^3 \vartheta_3^2 \vartheta_4 \vartheta_5}{\eta^6} = \mathrm{TB}_2[1,1,1,2,2,2,3,3,4,5].$$

- $\prod_{\ell \in [1,1,1,2,2,2,3,3,4,5]} \left(\zeta^{\ell/2} - \zeta^{-\ell/2}\right)$, the *baby* theta block.
- Given a theta block, it is easy to calculate the weight, index, character, divisor and valuation.

Skoruppa's Valuation

Definition

For $\phi \in J_{k,m}^{\mathrm{wh}}$, $x \in \mathbb{R}$, define $\mathrm{ord}(\phi; x) = \min_{(n,r) \in \mathrm{supp}(\phi)}(mx^2 + rx + n)$

$$\mathrm{ord} : J_{k,m}^{\mathrm{wh}} \to \text{Continuous piecewise quadratic functions of period one}$$

Theorem (Gritsenko, Skoruppa, Zagier)

Let $\phi \in J_{k,m}^{\mathrm{wh}}$. *Then* $\phi \in J_{k,m} \iff \mathrm{ord}(\phi; x) \geq 0$ *and* $\phi \in J_{k,m}^{\mathrm{cusp}} \iff \mathrm{ord}(\phi; x) > 0$.

Skoruppa's Valuation

Definition

For $\phi \in J^{\mathrm{wh}}_{k,m}$, $x \in \mathbb{R}$, define $\mathrm{ord}(\phi; x) = \min_{(n,r)\in \mathrm{supp}(\phi)}(mx^2 + rx + n)$

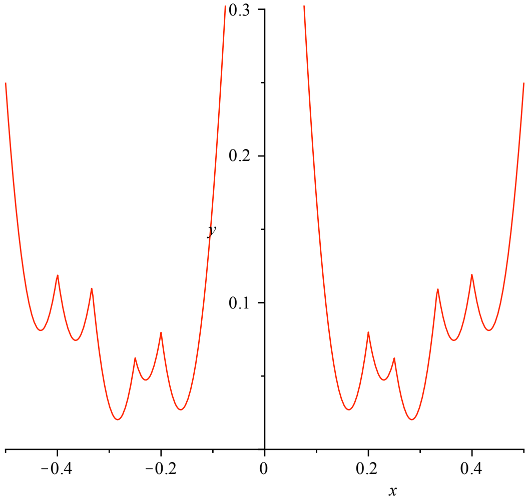
$$\mathrm{ord} : J^{\mathrm{wh}}_{k,m} \to \text{Continuous piecewise quadratic functions of period one}$$

Theorem (Gritsenko, Skoruppa, Zagier)

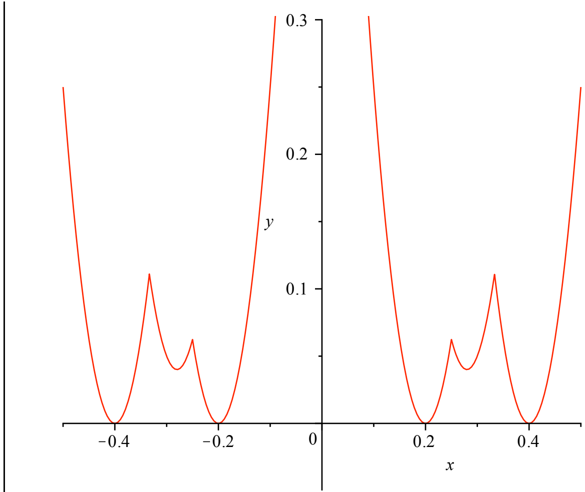
Let $\phi \in J^{\mathrm{wh}}_{k,m}$*. Then* $\phi \in J_{k,m} \iff \mathrm{ord}(\phi; x) \geq 0$ *and* $\phi \in J^{\mathrm{cusp}}_{k,m} \iff \mathrm{ord}(\phi; x) > 0$*.*

- $B_2(x) = x^2 - x - \frac{1}{6}$ and $\bar{B}(x) = B(x - \lfloor x \rfloor)$
- A lovely formula:

$$\mathrm{ord}\left(\mathrm{TB}_k[d_1, d_2, \ldots, d_\ell]\right) = \frac{k}{12} + \frac{1}{2}\sum_i \bar{B}_2(d_i x)$$



Cuspidal weight 2, index 37 theta block: [1, 1, 1, 2, 2, 2, 3, 3, 4, 5]



Jacobi Eisenstein weight 2, index 25 theta block:
[1, 1, 1, 1, 2, 2, 2, 3, 3, 4]

The shape of Theta Blocks to come

- A $\dfrac{10\vartheta}{6\eta}$ theta block has weight $10(\frac{1}{2}) - 6(\frac{1}{2}) = 2$.

The shape of Theta Blocks to come

- A $\dfrac{10\vartheta}{6\eta}$ theta block has weight $10(\frac{1}{2}) - 6(\frac{1}{2}) = 2$.
- A $\dfrac{10\vartheta}{6\eta}$ theta block has leading q-power $10(\frac{1}{8}) - 6(\frac{1}{24}) = 1$.

The shape of Theta Blocks to come

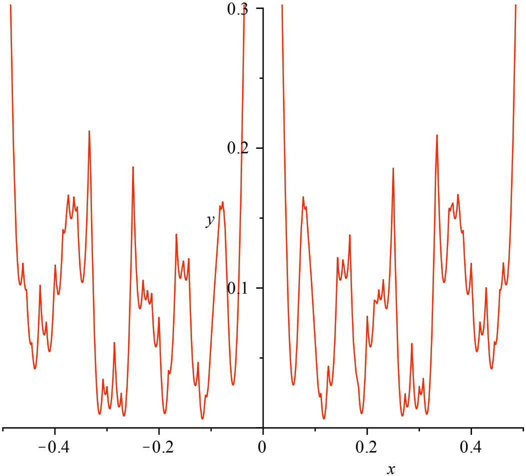
- A $\dfrac{10\vartheta}{6\eta}$ theta block has weight $10(\frac{1}{2}) - 6(\frac{1}{2}) = 2$.
- A $\dfrac{10\vartheta}{6\eta}$ theta block has leading q-power $10(\frac{1}{8}) - 6(\frac{1}{24}) = 1$.
- A $\dfrac{10\vartheta}{6\eta}$ theta block has index $m = \frac{1}{2}(d_1^2 + d_2^2 + \cdots + d_{10}^2)$.

The shape of Theta Blocks to come

- A $\dfrac{10\vartheta}{6\eta}$ theta block has weight $10(\frac{1}{2}) - 6(\frac{1}{2}) = 2$.
- A $\dfrac{10\vartheta}{6\eta}$ theta block has leading q-power $10(\frac{1}{8}) - 6(\frac{1}{24}) = 1$.
- A $\dfrac{10\vartheta}{6\eta}$ theta block has index $m = \frac{1}{2}(d_1^2 + d_2^2 + \cdots + d_{10}^2)$.
- Are there any other ways to get weight two?

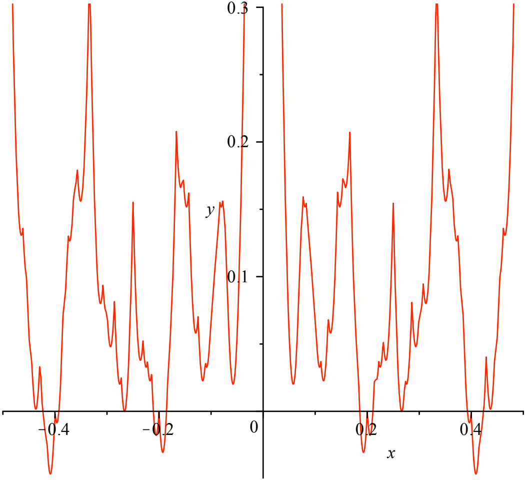
The shape of Theta Blocks to come

- A $\frac{10\vartheta}{6\eta}$ theta block has weight $10(\frac{1}{2}) - 6(\frac{1}{2}) = 2$.
- A $\frac{10\vartheta}{6\eta}$ theta block has leading q-power $10(\frac{1}{8}) - 6(\frac{1}{24}) = 1$.
- A $\frac{10\vartheta}{6\eta}$ theta block has index $m = \frac{1}{2}(d_1^2 + d_2^2 + \cdots + d_{10}^2)$.
- Are there any other ways to get weight two?
- A $\frac{22\vartheta}{18\eta}$ theta block has weight $22(\frac{1}{2}) - 18(\frac{1}{2}) = 2$.
- A $\frac{22\vartheta}{18\eta}$ theta block has leading q-power $22(\frac{1}{8}) - 18(\frac{1}{24}) = 2$.
- A $\frac{22\vartheta}{18\eta}$ theta block has index $m = \frac{1}{2}(d_1^2 + d_2^2 + \cdots + d_{22}^2)$.



Cuspidal weight 2, index 587 theta block:
[1, 1, 2, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 8, 8, 9, 10, 11, 12, 13, 14]

>
>



Weak weight 2, index 587 theta block:
[1, 1, 2, 2, 2, 3, 3, 4, 4, 5, 6, 6, 6, 7, 8, 8, 9, 10, 11, 12, 13, 14]

Index Raising Operators $V(\ell) : J_{k,m} \to J_{k,m\ell}$

$$\text{Elliptic Hecke Algebra} \longrightarrow \text{Jacobi Hecke Algebra}$$

$$\sum \mathrm{SL}_2(\mathbb{Z}) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \sum P_{2,1}(\mathbb{Z}) \begin{pmatrix} a & 0 & b & 0 \\ 0 & ad-bc & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sum_{\substack{ad=\ell \\ b \bmod d}} \mathrm{SL}_2(\mathbb{Z}) \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \sum_{\substack{ad=\ell \\ b \bmod d}} P_{2,1}(\mathbb{Z}) \begin{pmatrix} a & 0 & b & 0 \\ 0 & \ell & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T(\ell) \mapsto V(\ell)$$

Gritsenko Lift

Definition

For $\phi \in J^{\mathrm{wh}}_{k,m}$, define a series by

$$\mathrm{Grit}(\phi)\begin{pmatrix}\tau & z\\ z & \omega\end{pmatrix} = \sum_{\ell\in\mathbb{N}} \ell^{2-k}(\phi|V(\ell))(\tau,z)e(\ell m\omega).$$

Theorem (Gritsenko)

For $\phi \in J^{\mathrm{cusp}}_{k,m}$ the series $\mathrm{Grit}(\phi)$ converges and defines a map

$$\mathrm{Grit}: J^{\mathrm{cusp}}_{k,m} \to S_k\left(K(m)\right)^{\epsilon}, \quad \epsilon = (-1)^k.$$

Gritsenko Lift

Definition

For $\phi \in J^{\mathrm{wh}}_{k,m}$, define a series by

$$\mathrm{Grit}(\phi)\begin{pmatrix}\tau & z\\ z & \omega\end{pmatrix} = \sum_{\ell\in\mathbb{N}} \ell^{2-k}(\phi|V(\ell))(\tau,z)e(\ell m\omega).$$

Theorem (Gritsenko)

For $\phi \in J^{\mathrm{cusp}}_{k,m}$ the series $\mathrm{Grit}(\phi)$ converges and defines a map

$$\mathrm{Grit} : J^{\mathrm{cusp}}_{k,m} \to S_k\left(K(m)\right)^{\epsilon}, \quad \epsilon = (-1)^k.$$

- Example: $\mathrm{Grit}\left(\eta^{18}\vartheta^2\right) = X_{10} \in S_{10}(K(1))$

There are 10 dimensions of Gritsenko lifts in $S_2(K(277))$

We have $\dim S_2(K(277)) = 11$ whereas the dimension of Gritsenko lifts in $S_2(K(277))$ is $\dim J_{2,277}^{\text{cusp}} = 10$.

Let $G_i = \text{Grit}(\text{TB}_2(\Sigma_i))$ for $1 \le i \le 10$ be the lifts of the 10 theta blocks given by:

$$\Sigma_i \in \{ [2, 4, 4, 4, 5, 6, 8, 9, 10, 14], [2, 3, 4, 5, 5, 7, 7, 9, 10, 14], [2, 3, 4, 4, 5, 7, 8, 9, 11, 13], [2, 3, 3, 5, 6, 6, 8, 9, 11, 13], [2, 3, 3, 5, 5, 8, 8, 8, 11, 13], [2, 3, 3, 5, 5, 7, 8, 10, 10, 13], [2, 3, 3, 4, 5, 6, 7, 9, 10, 15], [2, 2, 4, 5, 6, 7, 7, 9, 11, 13], [2, 2, 4, 4, 6, 7, 8, 10, 11, 12], [2, 2, 3, 5, 6, 7, 9, 9, 11, 12] \}.$$

The nonlift paramodular eigenform $f_{277} \in S_2(K(277))$

$$f_{277} = \frac{Q}{L}$$

$$\begin{aligned}
Q = & -14G_1^2 - 20G_8G_2 + 11G_9G_2 + 6G_2^2 - 30G_7G_{10} + 15G_9G_{10} + 15G_{10}G_1 \\
& - 30G_{10}G_2 - 30G_{10}G_3 + 5G_4G_5 + 6G_4G_6 + 17G_4G_7 - 3G_4G_8 - 5G_4G_9 \\
& - 5G_5G_6 + 20G_5G_7 - 5G_5G_8 - 10G_5G_9 - 3G_6^2 + 13G_6G_7 + 3G_6G_8 \\
& - 10G_6G_9 - 22G_7^2 + G_7G_8 + 15G_7G_9 + 6G_8^2 - 4G_8G_9 - 2G_9^2 + 20G_1G_2 \\
& - 28G_3G_2 + 23G_4G_2 + 7G_6G_2 - 31G_7G_2 + 15G_5G_2 + 45G_1G_3 - 10G_1G_5 \\
& - 2G_1G_4 - 13G_1G_6 - 7G_1G_8 + 39G_1G_7 - 16G_1G_9 - 34G_3^2 + 8G_3G_4 \\
& + 20G_3G_5 + 22G_3G_6 + 10G_3G_8 + 21G_3G_9 - 56G_3G_7 - 3G_4^2, \\
L = & - G_4 + G_6 + 2G_7 + G_8 - G_9 + 2G_3 - 3G_2 - G_1.
\end{aligned}$$

Euler factors for $f_{277} \in S_2(K(277))$

$$
\begin{aligned}
L(f, s, \mathrm{spin}) = & \left(1 + 2x + 4x^2 + 4x^3 + 4x^4\right) \\
& \left(1 + x + x^2 + 3x^3 + 9x^4\right) \\
& \left(1 + x - 2x^2 + 5x^3 + 25x^4\right) \\
& \dots
\end{aligned}
$$

Euler factors for $f_{277} \in S_2(K(277))$

$$
\begin{aligned}
L(f, s, \mathrm{spin}) = &\left(1 + 2x + 4x^2 + 4x^3 + 4x^4\right) \\
&\left(1 + x + x^2 + 3x^3 + 9x^4\right) \\
&\left(1 + x - 2x^2 + 5x^3 + 25x^4\right) \\
&\cdots
\end{aligned}
$$

- These match the 2, 3 and 5 Euler factors for $L(\mathcal{A}_{277}, s, \mathrm{H\text{-}W})$
- $\mathcal{A}_{277}$ = Jacobian of $y^2 + y = x^5 + 5x^4 + 8x^3 + 6x^2 + 2x$

Euler factors for $f_{277} \in S_2(K(277))$

$$L(f, s, \mathrm{spin}) = \left(1 + 2x + 4x^2 + 4x^3 + 4x^4\right)$$
$$\left(1 + x + x^2 + 3x^3 + 9x^4\right)$$
$$\left(1 + x - 2x^2 + 5x^3 + 25x^4\right)$$
$$\ldots$$

- These match the 2, 3 and 5 Euler factors for $L(\mathcal{A}_{277}, s, \mathrm{H\text{-}W})$
- $\mathcal{A}_{277}$ = Jacobian of $y^2 + y = x^5 + 5x^4 + 8x^3 + 6x^2 + 2x$
- A spin L-function not of GL(2) type.

Joint work with V. Gritsenko

$S_2(K(587))^- = \mathbb{C}B$ is spanned by a Borcherds product B.

(A minus form in weight two cannot be a lift.)

Why did Gritsenko suspect that the first minus form might be a Borcherds product?

$$11 = \min\{p : S_2(\Gamma_0(p)) \neq \{0\}\}, \qquad S_2(\Gamma_0(11)) = \mathbb{C}\,\eta(\tau)^2\eta(11\tau)^2$$

$$37 = \min\{p : J_{2,p}^{\text{cusp}} \neq \{0\}\}, \qquad J_{2,37}^{\text{cusp}} = \mathbb{C}\,\eta^{-6}\vartheta_1^3\vartheta_2^3\vartheta_3^2\vartheta_4\vartheta_5$$

$$587 = \min\{p : S_2(K(p))^- \neq \{0\}\}, \qquad S_2(K(587))^- = \mathbb{C}\ \text{Borch}(\psi)$$

$$\psi \in J_{0,587}^{\text{wh}}(\mathbb{Z})$$

- Let's come to grips with Borcherds products.

Theorem (Borcherds, Gritsenko, Nikulin)

Let $N, N_o \in \mathbb{N}$. *Let* $\Psi \in J_{0,N}^{\mathrm{wh}}$ *be a weakly holomorphic Jacobi form with Fourier expansion*

$$\Psi(\tau, z) = \sum_{n,r \in \mathbb{Z}:\, n \geq -N_o} c\,(n, r)\, q^n \zeta^r$$

and $c(n, r) \in \mathbb{Z}$ *for* $4Nn - r^2 \leq 0$. *Then we have* $c(n, r) \in \mathbb{Z}$ *for all* $n, r \in \mathbb{Z}$. *We set*

$$24A = \sum_{\ell \in \mathbb{Z}} c(0, \ell); \quad 2B = \sum_{\ell \in \mathbb{N}} \ell c(0, \ell); \quad 4C = \sum_{\ell \in \mathbb{Z}} \ell^2 c(0, \ell);$$

$$D_0 = \sum_{n \in \mathbb{Z}:\, n<0} \sigma_0(-n) c(n, 0); \;\; k = \frac{1}{2} c(0, 0); \;\; \chi = (\epsilon^{24A} \times v_H^{2B}) \chi_F^{k+D_0}.$$

There is a function $\mathrm{Borch}(\Psi) \in M_k^{\mathrm{mero}}\left(K(N)^+, \chi\right)$ *whose divisor in* in $K(N)^+\backslash\mathcal{H}_2$ consists of Humbert surfaces $\mathrm{Hum}(T_o)$ for $T_o = \left(\begin{smallmatrix} n_o & r_o/2 \\ r_o/2 & Nm_o \end{smallmatrix}\right)$ with $\gcd(n_o, r_o, m_o) = 1$ and $m_o \geq 0$. The multiplicity of $\mathrm{Borch}(\Psi)$ on $\mathrm{Hum}(T_o)$ is $\sum_{n\in\mathbb{N}} c(n^2 n_o m_o, n r_o)$. In particular, if $c(n, r) \geq 0$ when $4Nn - r^2 \leq 0$ then $\mathrm{Borch}(\Psi) \in M_k\left(K(N)^+, \chi\right)$ is holomorphic. In particular,

$$\mathrm{Borch}(\Psi)(\mu_N\langle Z\rangle) = (-1)^{k+D_0}\,\mathrm{Borch}(\Psi)(Z), \text{ for } Z \in \mathcal{H}_2.$$

For sufficiently large λ, for $Z = \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in \mathcal{H}_2$ and $q = e(\tau)$, $\zeta = e(z)$, $\xi = e(\omega)$, the following product converges on $\{Z \in \mathcal{H}_2 : \mathrm{Im}\, Z > \lambda I_2\}$:

$$\mathrm{Borch}(\Psi)(Z) = q^A\zeta^B\xi^C \prod_{\substack{n,r,m\in\mathbb{Z}:\, m\geq 0,\ \text{if } m=0 \text{ then } n\geq 0 \\ \text{and if } m=n=0 \text{ then } r<0.}} \left(1 - q^n\zeta^r\xi^{Nm}\right)^{c(nm,r)}$$

and is on $\{\Omega \in \mathcal{H}_2 : \operatorname{Im}\Omega > \lambda I_2\}$ a rearrangement of

$$\operatorname{Borch}(\Psi) = \left(\eta^{c(0,0)} \prod_{\ell \in \mathbb{N}} \left(\frac{\tilde{\vartheta}_\ell}{\eta} \right)^{c(0,\ell)} \right) \exp\left(-\operatorname{Grit}(\Psi) \right).$$

Borcherds Product Summary

Theorem

So, somehow, if you have a weakly holomorphic weight zero, index N Jacobi form with integral coefficients

$$\Psi(\tau, z) = \sum_{n,r\in\mathbb{Z}:\, n\geq -N_o} c\,(n, r)\, q^n \zeta^r$$

and the "singular coeffients" $c(n, r)$ with $4Nn - r^2 < 0$ are for the most part positive, then

$$\mathrm{Borch}(\Psi)(Z)=q^A\zeta^B\xi^C \prod_{n,m,r} \left(1 - q^n\zeta^r\xi^{Nm}\right)^{c(nm,r)}$$

converges in a neighborhood of infinity and analytically continues to an element of $M_{k'}\,(K(N))$, for some new weight k'.

Borcherds Product Example

- $$\phi_{10} = \eta^{18}\vartheta_1^2 \in J_{10,1}^{\text{cusp}}$$

Borcherds Product Example

- $$\phi_{10} = \eta^{18}\vartheta_1^2 \in J_{10,1}^{\text{cusp}}$$
- $$\psi = -\frac{\phi_{10}|V(2)}{\phi_{10}} = \sum_{n,r\in\mathbb{Z}:\, n\geq 1} c\left(n,r;\psi\right)q^n\zeta^r \in J_{0,1}^{\text{weak}}$$
$$= 20 + 2\zeta + 2\zeta^{-1} + \ldots$$

Borcherds Product Example

- $$\phi_{10} = \eta^{18}\vartheta_1^2 \in J_{10,1}^{\text{cusp}}$$
- $$\psi = -\frac{\phi_{10}|V(2)}{\phi_{10}} = \sum_{n,r\in\mathbb{Z}:\ n\geq 1} c\,(n,r;\psi)\,q^n\zeta^r \in J_{0,1}^{\text{weak}}$$
$$= 20 + 2\zeta + 2\zeta^{-1} + \ldots$$
- $$X_{10} = \text{Borch}(\psi)(Z) = q\zeta\xi \prod_{n,m,r} (1 - q^n\zeta^r\xi^m)^{c(nm,r;\psi)}$$

Borcherds Product Example

- $$\phi_{10} = \eta^{18}\vartheta_1^2 \in J_{10,1}^{\text{cusp}}$$
- $$\psi = -\frac{\phi_{10}|V(2)}{\phi_{10}} = \sum_{n,r\in\mathbb{Z}:\, n\geq 1} c\,(n,r;\psi)\,q^n\zeta^r \in J_{0,1}^{\text{weak}}$$
  $$= 20 + 2\zeta + 2\zeta^{-1} + \ldots$$
- $$X_{10} = \mathrm{Borch}(\psi)(Z) = q\zeta\xi \prod_{n,m,r} (1 - q^n\zeta^r\xi^m)^{c(nm,r;\psi)}$$
- $$\mathrm{Div}\,(\mathrm{Borch}(\psi)) = 2\,\mathrm{Hum}\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} = 2\,\mathrm{Sp}_2(\mathbb{Z})(\mathcal{H}_1 \times \mathcal{H}_1)$$
- The reducible locus: $\mathrm{Sp}_2(\mathbb{Z})(\mathcal{H}_1 \times \mathcal{H}_1) \subseteq \mathrm{Sp}_2(\mathbb{Z})\backslash\mathcal{H}_2$

A nonlift Borcherds Product in $S_2(K(587))^-$

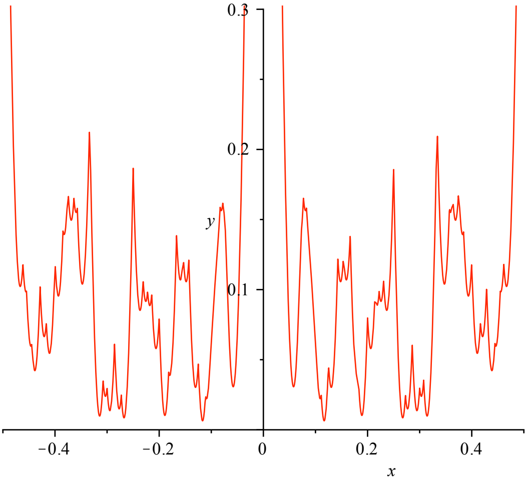
- Want: antisymmetric B-product $f \in S_2(K(p))^-$, here $p = 587$.
- Fourier Jacobi expansion: $f = \phi_p \xi^p + \phi_{2p} \xi^{2p} + \dots$
- ϕ_p is a theta block because f is a B-prod.
- $\phi_p \sim q^2$ because f is antisymmetric

A nonlift Borcherds Product in $S_2(K(587))^-$

- Want: antisymmetric B-product $f \in S_2(K(p))^-$, here $p = 587$.
- Fourier Jacobi expansion: $f = \phi_p \xi^p + \phi_{2p} \xi^{2p} + \dots$
- ϕ_p is a theta block because f is a B-prod.
- $\phi_p \sim q^2$ because f is antisymmetric

- The only element of $J_{2,587}^{\mathrm{cusp}}$ that vanishes to order two is:

$$\mathrm{TB}_2 \boxed{2} =$$
$$\mathrm{TB}_2[1,1,2,2,2,3,3,4,4,5,5,6,6,7,8,8,9,10,11,12,13,14]$$



Cuspidal weight 2, index 587 theta block:
[1, 1, 2, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 8, 8, 9, 10, 11, 12, 13, 14]

The Ansatz

Maybe this will work.

Ansatz

Define a Theta Buddy $\Theta \in J_{2,2\cdot 587}^{\text{cusp}}$ *by*

$$\phi_{2p} = \phi_p | V(2) - \Theta$$

The Ansatz

Maybe this will work.

Ansatz

Define a Theta Buddy $\Theta \in J^{\text{cusp}}_{2,2\cdot 587}$ *by*

$$\phi_{2p} = \phi_p | V(2) - \Theta$$

- By antisymmetry and the action of $V(2)$

$$\operatorname{coef}(q^2, \Theta) = \operatorname{coef}(q^4, \phi_p) = \prod_{\ell \in \boxed{3}} \left(\zeta^{\ell/2} - \zeta^{-\ell/2} \right)$$

The Ansatz

Maybe this will work.

Ansatz

Define a Theta Buddy $\Theta \in J^{\text{cusp}}_{2,2\cdot 587}$ *by*

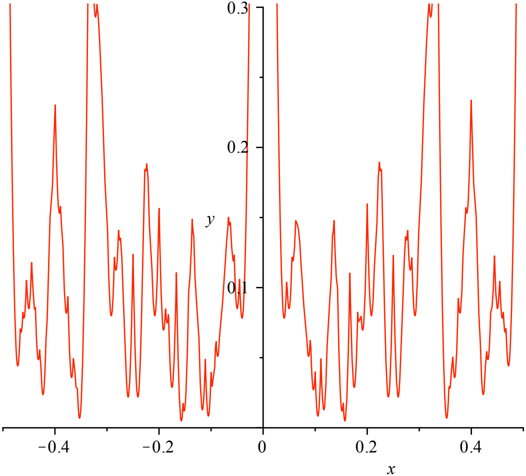
$$\phi_{2p} = \phi_p | V(2) - \Theta$$

- By antisymmetry and the action of $V(2)$

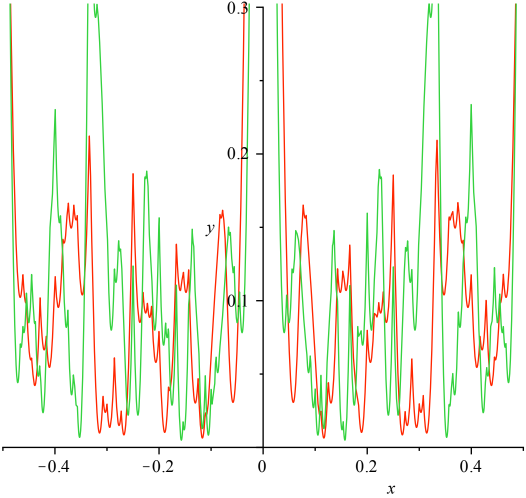
$$\text{coef}(q^2, \Theta) = \text{coef}(q^4, \phi_p) = \prod_{\ell \in \boxed{3}} \left(\zeta^{\ell/2} - \zeta^{-\ell/2}\right)$$

- The leading coefficient of the Theta Buddy is a Baby Theta Block:

$$\Theta = \text{TB}_2 \boxed{3} =$$
$$\text{TB}_2[1, 1, 2, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 8, 8, 9, 10, 11, 12, 13, 14]$$



Cuspidal weight 2, index 1174 theta block:
[1, 10, 2, 2, 18, 3, 3, 4, 4, 15, 5, 6, 6, 7, 8, 16, 9, 10, 22, 12, 13, 14]



Cuspidal weight 2, index 587 and 1174 theta blocks:

>

>

- Define

$$\psi = \frac{\mathrm{TB}_2\boxed{2}\,|V(2) - \mathrm{TB}_2\boxed{3}}{\mathrm{TB}_2\boxed{2}} \in J_{0,587}^{\mathrm{wh}}$$

$$= 4 + \frac{1}{q} + \zeta^{-14} + \cdots + q^{134}\zeta^{561} + \cdots$$

- Define

$$\psi = \frac{\mathrm{TB}_2\boxed{2}\,|V(2) - \mathrm{TB}_2\boxed{3}}{\mathrm{TB}_2\boxed{2}} \in J^{\mathrm{wh}}_{0,587}$$
$$= 4 + \frac{1}{q} + \zeta^{-14} + \cdots + q^{134}\zeta^{561} + \cdots$$

- Compute the singular part of ψ to order $q^{146} = q^{\lfloor p/4 \rfloor}$ and see that all singular Fourier coefficients $c(n, r; \psi) \geq 0$.

- Define

$$\psi = \frac{\mathrm{TB}_2\boxed{2}\,|V(2) - \mathrm{TB}_2\boxed{3}}{\mathrm{TB}_2\boxed{2}} \in J^{\mathrm{wh}}_{0,587}$$
$$= 4 + \frac{1}{q} + \zeta^{-14} + \cdots + q^{134}\zeta^{561} + \cdots$$

- Compute the singular part of ψ to order $q^{146} = q^{\lfloor p/4 \rfloor}$ and see that all singular Fourier coefficients $c(n, r; \psi) \geq 0$.
- Therefore, $\mathrm{Borch}(\psi) \in S_2\left(K(587)\right)^-$ exists and hence spans a one dimensional space.

- Compute the 2 and 3-Euler factors

$$
\begin{aligned}
L(f, s, \mathrm{spin}) = &\left(1 + 3x + 9x^2 + 6x^3 + 4x^4\right) \\
&\left(1 + 4x + 9x^2 + 12x^3 + 9x^4\right) \\
&\ldots
\end{aligned}
$$

- Compute the 2 and 3-Euler factors

$$L(f,s,\mathrm{spin}) = \left(1+3x+9x^2+6x^3+4x^4\right)\\ \left(1+4x+9x^2+12x^3+9x^4\right)\\ \cdots$$

- These match the 2 and 3 Euler factors for $L(\mathcal{A}_{587}^{-}, s, \mathrm{H\text{-}W})$
- $\mathcal{A}_{587}^{-}$ = Jacobian of $y^2 + (x^3+x+1)y = -x^3 + -x^2$

Current Work

We are using Borcherds products to construct more paramodular nonlifts.

Thank you!