MINIMUM DISCRIMINANTS OF IMPRIMITIVE DECIC FIELDS

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Abstract. In this paper, we compute the minimum discriminants of imprimitive degree 10 fields for different combinations of Galois group and signature. We use class field theory when there is a quintic subfield, and a Martinet search in the more difficult case where there is only a quadratic subfield.

If L is a degree n number field, let D_L be its discriminant, and $d_L := |D_L|$ its absolute discriminant. Let $G(L)$ be a transitive subgroup of S_n giving the Galois group of the Galois closure of $L/\mathbb{Q}$, which is well-defined up to conjugation. We let r_1 denote the number of real places of L.

For a given positive integer n and group $G \leq S_n$, several authors have contributed to our knowledge of minimizing d_L for fields L with $[L : \mathbb{Q}] = n$ and $G(L) = G$. In [Klüners and Malle 2001], Klüners and Malle discuss the more refined problem of minimizing d_L with n, G, and r_1 fixed, as well as the still more refined analogue of this question where the conjugacy class of complex conjugation specified. Their results are primarily for degrees ≤ 8, and there have been some results in degree 9. Here we consider imprimitive fields in degree $n = 10$ and attempt to determine minimal examples for fixed G and r_1.

The computation for imprimitive decics divides naturally into three cases, which we label as follows. We refer to decics L which contain a quadratic but not quintic subfield as Type 2 fields, those which contain a quintic but no quadratic subfield as Type 5 fields, and those which contain both as Type 2-5 fields. The most difficult cases are Type 2 fields where we employed computer searches based on Martinet's generalization of Hunter's theorem. Some of these were standard Martinet searches, and some were targeted Martinet searches in the sense of [Driver and Jones 2009]. This is explained in Section 1.

Section 2 treats Type 5 decics using class field theory. These computations rely on having a sufficiently large complete list of quintic fields to work from. We extended the range of known totally real quintic fields K from $d_K \leq 2 \cdot 10^7$ to $d_K \leq 10^8$ which in turn, allowed us to determine more minimal decic discriminants.

Finally, in Section 3 we describe results for the simplest case, Type 2-5 decic fields. These fields are just tensor products of their subfields, so working with tables of lower degree fields suffices here. Again, we employed our extended list of totally real quintic fields here.

1. Type 2 Decics

1.1. **The Martinet Search.**
A standard technique for searching for all fields with absolute discriminant less than a given bound comes from the geometry of numbers. In particular, all such primitive degree n extensions of a degree m field can in principle be found using Martinet's theorem, a generalization of Hunter's theorem [Cohen 2000, Sec 9.3].

Here, we use Martinet's theorem to find all quintic extensions of a quadratic base field. If a decic field L with a quadratic subfield K satisfies the discriminant bound $d_L \leq B$, then the discriminant of K must satisfy

$$d_K \leq d_L^{1/5} \leq B^{1/5}.$$

In this paper, we take $B = 10^{10}$. Thus, it is necessary to search over each quadratic base field satisfying $d_K \leq 100$; there are 61 such quadratic fields.

In [Driver and Jones 2009], we show how to target particular discriminants in a Martinet search. Note, our search here is naturally broken up into 61 searches, one for each quadratic field. When the quadratic base field has relatively large discriminant there are correspondingly few possibilities for the decic discriminant beneath the bound B. For example, when the quadratic base field is $K = \mathbb{Q}(\sqrt{97})$, the only possibility for the decic field is $d_L = 97^5$.

When we have a priori knowledge of the absolute decic discriminant, we can first test to make sure that the ratio of the absolute polynomial discriminant to d_L is a square. This usually weeds out 99% of the candidate polynomials, and allows us to skip the more computationally expensive tasks of checking for irreducibility and computing field discriminants. Even when multiple discriminants are possible, it is usually much faster to use several targeted searches instead of a single standard search.

The use of targeting is only helpful here for those base fields having the largest discriminant bounds; as the discriminant of the base field drops, a point is reached where too much targeting would be required in order to be practical. In our implementation, we found that a good rule of thumb is to use a targeted Martinet search for those base fields having $d_K \geq .75B^{1/5}$.

1.2. **Results.**
For our first attempt, we used a discriminant bound of $B = 10^9$. The Martinet search took approximately one week on a 3GHz Pentium, and found 239 fields.

To extend the search further, we used a distributed computing approach through the BOINC system [BOINC]. With $B = 10^{10}$, the computation took a total of approximately 20,000 machine hours (summed over all hosts). It found 10,565 fields. Table 1 shows the distribution of fields partitioned by type and signature. If no fields of a particular Galois group (resp. signature)

Table 1: Numbers of decics L with a quadratic subfield and $d_L \leq 10^{10}$.

Gal	**Signature** (0,5)	(2,4)	(4,3)	**Total**
$T1$	1			1
$T2$	2			2
$T3$	6	2		8
$T5$	1	0		1
$T6$	6			6
$T11$	2	0		2
$T21$	14	12		26
$T22$	158	14		172
$T33$	1	0		1
$T40$	2	4		6
$T41$	5	0		5
$T43$	9187	946	202	10335

were found then the corresponding row (resp. column) was left out of the table. Impossible combinations of type and signature are grayed out.

Table 2 gives the minimum discriminants for Type 2 decics as determined

Table 2: Minimum discriminants and defining polynomials of Type 2 decics.

Gal	r_1	**Discrim.**	**Polynomial**
$T6$	0	$-3^5 31^4$	$x^{10}+2x^8-3x^7+3x^6-7x^5+8x^4-7x^3+7x^2-4x+1$
$T21$	0	$-2^{10} 761^2$	$x^{10}-4x^9+8x^8-10x^7+9x^6-6x^5+3x^4-x^2+1$
$T21$	2	$5^5 601^2$	$x^{10}-3x^9+5x^8-5x^7+2x^6+2x^5-4x^4+2x^3-2x+1$
$T33$	0	$-3^5 337^3$	$x^{10}-4x^9+7x^8-11x^7+18x^6-19x^5+16x^4-17x^3+10x^2-3x+3$
$T40$	0	$-3^5 4903^2$	$x^{10}-x^9+5x^8-3x^7+8x^6-4x^5+4x^4-3x^3-x+1$
$T40$	2	$5^5 1429^2$	$x^{10}-4x^9+8x^8-12x^7+18x^6-21x^5+18x^4-9x^3+x^2+2x-1$
$T41$	0	$-2^6 3^5 13^2 43^2$	$x^{10}-x^9-4x^8+x^7+8x^6+2x^5-6x^4-x^3+7x^2+5x+1$
$T43$	0	$-3^5 11^2 7369$	$x^{10}-3x^9+5x^8-6x^7+6x^6-4x^5+2x^4-x^3+x^2-x+1$
$T43$	2	$5^5 280001$	$x^{10}-x^9-x^8+3x^7-3x^6-x^5+5x^4-x^3-3x^2+x+1$
$T43$	4	$-5^5 1039^1 1049$	$x^{10}-3x^9+5x^8-4x^7-2x^6+8x^5-8x^4+x^3+3x^2-3x+1$

by our computer search. Since these polynomials were computationally the most difficult to obtain, we include defining polynomials of the corresponding fields. Here, signatures are given by just r_1, the number of real places. Discriminants are given in factored form. Here, and in subsequent tables, we include exponents of 1 in a factorization if needed to separate prime factors.

2. Type 5 Decics

2.1. **Totally Real Quintics.** Studying Type 5 decics relies on our knowledge of quintic fields. In the case of totally real quintics K, those with $d_K \leq 2 \cdot 10^7$ were classified by [Schwarz, Pohst, and Diaz Y Diaz 1994], and are available at [Bordeaux]. While the bound for d_K is large compared to the corresponding bounds for the two other signatures of quintic fields, it only accounts for 22,740 fields which we found insufficient.

We implemented a standard Hunter's search using the methods described in [Cohen 2000, §9.3] to find all totally real quintic fields K with $d_K \leq 10^8$. The number of fields broken down by Galois group is given in Table 3. Here we use common names for the Galois groups, which correspond to

Table 3: Numbers of totally real quintic number fields K with $d_K \leq 10^8$, partitioned by Galois group.

Gal	C_5	D_5	F_5	A_5	S_5
#	6	72	59	80	162,022

T1–T5 respectively using the labelling of [Butler and McKay 1983]. The fields themselves are available at [Jones and Roberts]. As was the case in [Schwarz, Pohst, and Diaz Y Diaz 1994], almost all of the fields have Galois group S_5. Although this is a fairly modest increase in the range of discriminants, it was of great use in computing the minimal absolute discriminants totally real decics of Type 5 and Type 2-5.

2.2. **Class Field Theory.** For decic fields containing a quintic subfield, class field theory provides a mechanism for computing the desired fields. Here, computations are relatively simple in comparison to more general cases of Kummer theory since our fields always contain $\zeta_2 = -1$, so the desired extensions can be found by taking square roots of appropriate elements. An algorithm for this is given in [Cohen 2000, §9.2.2].

All computations here were carried out with `gp` [PARI2]. We needed to deviate from the approach given in [Cohen 2000] because of memory issues. Given a base field K and bound B, a quadratic extension corresponding to modulus $(\mathfrak{m}_0, \mathfrak{m}_\infty)$ needs to satisfy $d_K^2 N(\mathfrak{m}_0) \leq B$. In some cases, it was impractical to first compute all $\mathfrak{m}_0$ with $N(\mathfrak{m}_0) \leq B/d_K^2$. Instead, we compute the list $\mathcal{L}$ of all such moduli which are only divisible by primes $p \leq B_1 := \sqrt{B/d_K^2}$. All additional moduli are then of the form $\mathfrak{m}_0 Q$ where

Q divides a prime q with $B_1 < q \leq B/d_K^2$, and $\mathfrak{m}_0$ from $\mathcal{L}$. So, as we compute these, we compute the corresponding fields as we go and do not have to store these additional moduli.

For example, consider the totally real S_5 quintic field of minimal discriminant. It has discriminant $61 \cdot 397$. With $B = 10^{16}$, the first phase found and stored the 766,157 moduli which are divisible only by primes less than $B_1 = 4129$. The total number of moduli was 3,851,600.

One could compute decic overfields of quintics using a Martinet search, as was done in [Selmane 2000a, Selmane 2000b, Selmane 2002]. We found, however, that class field theory was much more efficient.

2.3. **Results.** For complete searches of Type 5 decics, we computed all fields L with $d_L \leq 10^{12}$. When the quintic subfield was totally real, we computed all decics with $d_L \leq 10^{13}$. The results are summarized in Table 4. These

Table 4: Numbers of Type 5 decic fields L with $d_L \leq B$ by signature. Columns correspond to the number of real places for the decic field, and rows correspond to the number of real places of the quintic subfield.

r_1	B	0	2	4	6	8	10
1	10^{12}	226764	210453				
3	10^{12}	11972	37249	32852	7876		
5	10^{13}	772	3954	7527	7592	2849	313

computations include, and extend, those performed in [Selmane 2000a, Selmane 2000b, Selmane 2002]. We found one discrepancy with these prior computations, namely we found one more totally complex decic field with a quintic subfield with $r_1 = 3$ and $d_L \leq 10^{11}$. Otherwise, our results are consistent with [Selmane 2000a, Selmane 2000b, Selmane 2002].

Table 5 gives our findings for minimal absolute discriminants for Type 5 decics. Most entries come from the searches summarized above, namely from fields with $d_L \leq 10^{12}$. Almost all of the cases not settled by that search had a totally real quintic subfield. This naturally includes totally real decics ($r_1 = 10$), but also cases where $(T, r_1) = (16, 2)$, $(16, 6)$, $(24, 6)$, $(25, 6)$, $(29, 4)$, $(29, 6)$, $(29, 8)$, $(34, 6)$, $(36, 4)$, $(36, 6)$, $(36, 8)$, $(38, 6)$.

For example, consider a $T16$ decic L with $r_2 = 2$. The group $T16$ has two conjugacy classes of elements of order 2 and cycle type $2^2 1^6$. The unique quintic subfield K has $G(K) = D_5$, and in the projection $T16 \to D_5$ induced by the inclusion $K \hookrightarrow L$, elements of cycle type $2^2 1^6$ map to the identity. So, for a $T16$ decic with $r_1 = 2$, the quintic subfield is a totally real D_5 field.

In these cases, we made use of the quintics computed in Section 2.1. In each case, we computed decic overfields of the desired signature, where the quintic had a the correct Galois group, and $d_L \leq 10^{16}$. In this computation,

Table 5: Minimum discriminants for Type 5 decics.

Gal	**Signature**					
	$(0,5)$	$(2,4)$	$(4,3)$	$(6,2)$	$(8,1)$	$(10,0)$
$T8$		$11^8 23^2$		$11^8 23^2$		$2^{10} 11^8 23^2$
$T14$	$-11^8 1451$	$11^8 89$	$-11^8 23$	$11^8 109$	$-11^8 331$	$11^8 43^1 67$
$T15$		$2^{10} 47^4$		$3^2 401^4$		$67^2 410^4$
$T16$	$-5^2 47^5$	$3^2 401^5$		401^5		$17^2 401^5$
$T23$	$-47^4 83$	$47^4 193$	$-3^1 401^4$	$13^1 401^4$	$-3^3 401^4$	$397^1 401^4$
$T24$		$2^{16} 13^6$		$2^{10} 5^{10} 7^4$		?
$T25$		$2^{12} 13^7$		$2^{10} 5^{11} 7^4$		?
$T29$	$-2^8 13^6 31$	$2^8 13^6 41$	$-2^8 3^1 53^6$	$2^8 5^1 53^6$	$-2^8 19^1 53^6$	$2^8 5^{10} 7^4 97$
$T34$		?		$3^4 883^4$		$2^6 11^4 113^4$
$T36$	$-2^{12} 17^4 43$	$2^{14} 11^1 17^4$	$-2^4 3^1 881^4$	$2^6 17^1 881^4$	$-3^5 883^4$	$2^4 11^4 29^1 113^4$
$T37$		$5^2 5783^2$		$2^{10} 13^2 347^2$		$2^{10} 36497^2$
$T38$	$-3^8 223^3$	$2^8 227^3$	$-2^8 13^4 19^3$	$61^3 397^3$		$2^4 33769^3$
$T39$	$-7^2 23^1 431^2$	$7^2 41^1 631^2$	$-7^2 43^1 1217^2$	$461^1 5783^2$	$-61^2 167^1 397^2$	$61^2 397^2 757$

we only kept track of fields of minimal discriminant for each Galois group since the number of fields in some cases was so large.

3. Type 2-5 Decics

The simplest case is where the decic contains both a quadratic and quintic subfield. There are 8 corresponding decic Galois groups. These fields are easily generated from tables of quadratics and quintics by forming composita. We used quintics from the Bordeaux ftp site [Bordeaux], augmented by the results of our search for totally real quintics described in Section 2.1 above. Naturally, some care must be exercised to insure that one has searched far enough. Using this method, we were able to find the minimum discriminants for all the above mentioned decics, and for almost all possible signatures, with the results given in Table 6.

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Table 6: Minimum discriminants of Type 2–5 decics.

	Signature			
Gal	$(0,5)$	$(2,4)$	$(6,2)$	$(10,0)$
$T1$	-11^9			$3^5 11^9$
$T2$	-47^5			401^5
$T3$	$-3^5 47^4$	$7^4 17^5$		$5^5 401^4$
$T4$		$2^8 5^{11}$		$2^8 5^{11} 7^4$
$T5$	$-3^5 17^6$	$2^{15} 5^{10}$		$2^{15} 7^4 13^6$
$T11$	$-2^{16} 17^4$	$5^5 53^4$		$3^8 5^5 73^4$
$T12$	$-2^{10} 89^4$	$2^8 3^8 5^7$		$2^8 13^7 17^4$
$T22$	$-3^5 1123^2$	$5^5 13^2 41^2$	$13^5 347^2$	$5^5 61^2 397^2$

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