Iwasawa L-functions for Multiplicative Abelian Varieties

John W. Jones

Department of Mathematics
University of Texas
Austin, Texas 78712

Bernardi, Goldstein and Stephens ([1, 2]), and subsequently Mazur, Tate and Teitelbaum ([9]), have formulated p-adic analogues of Birch and Swinnerton-Dyer's conjectures for analytic p-adic L-functions of elliptic curves based on calculations of their special values and their derivatives. The calculations in [9] included curves with split multiplicative reduction at p, in which case the p-adic L-function appeared to vanish to higher order than their classical analogues.

One expects to find similar behavior between analytic p-adic L-functions and Iwasawa L-functions. Investigations of this type were initiated Perrin-Riou ([12]) and generalized by Schneider ([14, 15]). Their work was concerned with curves with good ordinary reduction. In this paper, we deal with the case of multiplicative reduction (thus encompassing the extra zeros).

The Iwasawa L-function of an elliptic curve is traditionally defined to be the characteristic polynomial of the p-Selmer group for the curve. We investigate this polynomial and find that while its leading coefficient has the predicted p-adic ordinal, it does not exhibit the extra zero suggesting that it is not the correct choice of Λ-module. We suggest several possibilities for the Λ-module which do have the correct order of vanishing and first non-vanishing coefficient. Finally, we prove a functional equation for these modules, generalizing the functional equation proved by Mazur ([7]) in the good ordinary reduction case.

I would like to thank my advisor Barry Mazur for his help and guidance.

1 Notations, Conventions and Terminology

We will observe the following notations, conventions and terminology throughout:

Let $p \neq 2$ be a prime number and let K be a number field. If F is a field, $\mathcal{O}_F$ will be its ring of integers (if appropriate).

A *quasi-isomorphism* is a group homomorphism with finite kernel and cokernel. Similarly, we will refer to *quasi-injections*, *quasi-surjections*, and *quasi-exact sequences* with the obvious meanings intended.

If G is an abelian group, we let G^* denote the Pontryagin dual of G; $\mathrm{Tor}\,G :=$ the torsion subgroup of G; $\mathrm{Div}\,G :=$ its subgroup of divisible elements; $G_{\mathrm{Tor}} :=$ $G/\mathrm{Tor}\,G$ and $G_{\mathrm{Div}} := G/\mathrm{Div}\,G$. Furthermore, if $f\colon A \to B$ is a homomorphism, we let $\mathrm{Div}\,f$ and f_{Div} be the induced maps $\mathrm{Div}\ A \to \mathrm{Div}\ B$ and $A_{\mathrm{Div}} \to B_{\mathrm{Div}}$ respectively. We follow the analogous convention for $\mathrm{Tor}\,f$ and f_{Tor}. We will use simply G to denote the order of G, except where confusion may result in which case we will use $\#G$.

If G is an abelian group, an abelian group scheme or a sheaf of abelian groups, G_n will denote the kernel of multiplication by n on G. We then let $G_{p^\infty} := \varinjlim G_{p^m}$ and $T_p(G) := \varprojlim G_{p^m}$.

The additive topological group $\mathbf{Z}_p$ will be denoted by Γ. The closed subgroup of index p^n will be written Γ_n.

Let L be a Γ-extension of K (i.e., an extension such that $Gal(L/K) \cong \Gamma$). An example of such an extension is the *cyclotomic* extension of K, the unique Γ-extension contained in $K(\boldsymbol{\mu}_{p^\infty})$, (the field gotten by adjoining all of the p^{th}-power roots of unity to K). We further assume that every prime of K is either unramified or totally ramified in L. Corresponding to L/K is a p-adic logarithm denoted $\log_p$. If L is the cyclotomic extension, then $\log_p$ is Iwasawa's logarithm.

Fix a continuous character

$$\kappa\colon \mathrm{Gal}(L/K) \to 1 + p^e\mathbf{Z}_p \subset \mathbf{Z}_p^*$$

such that the following diagram is commutative

$$\begin{array}{ccc} K_v^* & \xrightarrow{\text{rec}} & \Gamma \\ \downarrow{\scriptstyle \log_p} & & \downarrow{\scriptstyle \kappa} \\ \mathbf{Z}_p & \xleftarrow{\log_p} & \mathbf{Z}_p^* \end{array}$$

If L/K is the cyclotomic extension, then κ is the cyclotomic character (see [13] p. 402).

If t is a place of K, we will denote the completion of K at t by K_t. If L is a Γ-extension of K and t is a place which ramifies in L, then we define L_t to be the union of the completions at t of the finite intermediate extensions. We will simply write Γ for both $Gal(L_t/K_t)$ and $Gal(L/K)$. The set of all the primes of K ramifying in L will be denoted by T.

If a and b are two elements of a local field, we write $a \approx b$ if a/b is a unit.

If G is a commutative group scheme, and E/K is a finite Galois extension of fields, then we denote the norms of the points of the group scheme

$$\mathrm{Norm}_{E/K}G := \{ \sum_{g \in \mathrm{Gal}(E/K)} g\,x \quad \text{for} \quad x \in G(E)\}$$

where the sum is taken under the group law of G.

The *universal norms* of G for a Γ-extension L/K is the intersection of the groups of norms from each finite subextension E

$$NG(K) := \bigcap_{K \subset E \subset L} \mathrm{Norm}_{E/K}G.$$

Let $A_{/K}$ be an abelian variety over K, and A its Néron model over $\mathcal{O}_K$. The Néron model of the dual abelian variety of A is denoted $\tilde{A}$.

We let A^0 be the "connected component" of A. One then has the short exact sequence of sheaves for the *fpqf* site

$$0 \to A^0 \to A \to \Phi \to 0.$$

The restriction of Φ to the small etale site is a skyscraper sheaf composed on each fiber by the group of connected components for the reduction of A at that place. That is,

$$\Phi = \bigoplus_{v \in \mathcal{O}_K} {}_v\Phi$$

where all but finitely many of the ${}_v\Phi$ are trivial. The *fpqf* cohomology of Φ can be computed in the etale site by a theorem of Grothendieck since it is the quotient of smooth group schemes.

The image of multiplication by powers of p in A will stabilize for high powers. It is denoted by $\bar{A}$. If ν is sufficiently large and A is semi-stable at all primes dividing p, then there are short exact sequences for the *fpqf* site

$$0 \to \bar{A} \to A \to \Phi/p^\nu\Phi \to 0$$

and

$$0 \to A_{p^\nu} \to A \xrightarrow{p^\nu} \bar{A} \to 0. \tag{1}$$

For each of these constructions, A, A_{p^∞}, A^0, $\bar{A}$, Φ, we will not usually explicitly state to what the base field they correspond, as it should be clear from the context. In the case of A^0, there is little confusion possible since A^0 base changes properly if A is semi-stable. While we are primarily interested in elliptic curves with multiplicative reduction and the cyclotomic Γ-extension, we will use the following hypotheses on A, L and K.

1. $Ш_{p^\infty}(K)$ is finite;
2. every prime of K for which A has bad reduction splits finitely in L;
3. the reduction of A is semi-stable at every place of K dividing p, and is an extension of an ordinary abelian variety by a torus for every $t \in T$;
4. for every place $t \in T$, the universal norms of $A(L_t)$ are of finite index in $A(K_t)$.

The first condition is conjectured to always be true. The second is true for the cyclotomic extension of K. The last condition is known to hold in the following situations

- A has good ordinary reduction at v (see [7]);
- A is an elliptic curve with non-split multiplicative reduction at v (see [11]);

- A is an elliptic curve with split multiplicative reduction at v and $\log_p q \neq 0$, where $\log_p$ is a logarithm associated to the Γ-extension L/K, and q is the Tate parameter of A at v. This is, conjecturally, always the case (see [9]) for the cyclotomic extension.

In the reduction of A at place dividing p, the toroidal part is not necessarily split (isomorphic to a product of copies of $\mathbf{G}_m$). It does, however, contain a maximal split sub-torus. Let e_v be its dimension. We then let $e := \Sigma_{v \in T}\, e_v$.

Cohomology groups will be taken for the (big) *fpqf* site unless otherwise noted. The category of abelian sheaves for this site over a scheme X will be denoted by $Sh(X)$. If $X = spec(R)$, then we will write $Sh(R)$. Similarly, if R is a ring, $H^i(spec(R), F)$ will be denoted simply by $H^i(R, F)$.

If R is a local ring, then we denote the relative cohomology over $spec(R)$ with values in a sheaf F by $H^i_\cdot(R, F)$.

We will use equivariant cohomology as introduced in the appendix to [14]. The definitions and basic properties are given there. If E is an extension field of K, we will denote the equivariant cohomology for the extension of the respective integers with coefficients in a sheaf F by $H^i(\mathcal{O}_E/\mathcal{O}_K, F)$.

1.1 Λ-Modules

Some of the objects of principal interest will be finitely generated modules over the completed group ring

$$\Lambda := \varprojlim \mathbf{Z}_p[\Gamma/p^n\Gamma].$$

If we choose a topological generator $\gamma \in \Gamma$, then the map which sends T to $\gamma - 1$ induces an isomorphism from $\mathbf{Z}_p[[T]]$ to Λ.

A finitely generated Λ-module, M, is quasi-isomorphic to a Λ-module which is the direct sum of a free part Λ^ρ and a torsion part

$$\bigoplus_i \mathbf{Z}/p^{\mu_i}\mathbf{Z}[[T]] \bigoplus \bigoplus_j \mathbf{Z}_p[[T]]/(F_j)^{n_j}$$

where each F_j is an irreducible distinguished polynomial. (A distinguished polynomial is one which is congruent to T^n mod p where n is the degree of the polynomial). The invariants ρ, the μ_i, and the $F_j^{n_j}$ together completely describe M up to quasi-isomorphism. If $\rho = 0$, then the Λ-module is *torsion*. Finally, the *μ-invariant* of M is $\mu := \sum \mu_i$.

If M is a finitely generated torsion Λ-module, then its *characteristic polynomial* is given by

$$F_M(T) := p^\mu \prod_j (F_j(T))^{n_j}.$$

If M is not torsion, we set $F_M(T) := 0$.

Sometimes, one considers p^μ times the characteristic polynomial of γ acting on M instead. We denote this by $f_M(t)$. Clearly, the two polynomials are related by $f_M(T+1) = F_M(T)$.

We will be interested in the order of vanishing m of $F_M(T)$ at 0, and its first non-zero coefficient

$$F_M^{(m)}(0) := \frac{1}{m!}\frac{d^m}{dT^m}F_M(T)\mid_{T=0}$$

up to a p-adic unit.

The basic theory of Λ-modules provides us with the following.

Proposition 1.1

- $$m \geq \mathrm{rank}_{\mathbf{Z}_p} M^\Gamma.$$
- *If M is torsion, then the following are equivalent:*
    1. $$m = \mathrm{rank}_{\mathbf{Z}_p} M^\Gamma;$$
    2. *The representation of Γ on M is semi-simple on the 1-eigenspace;*
    3. *For each F_j which equals T, $n_j = 1$;*
    4. *The map* $$M^\Gamma \xrightarrow{g_M} M_\Gamma$$ *is a quasi-isomorphism.*
- *If the above equivalent conditions hold, then* $$m = \mathrm{rank}_{\mathbf{Z}_p} M^\Gamma = \mathrm{rank}_{\mathbf{Z}_p} M_\Gamma$$ *and* $$F_M^{(m)}(0) = \mathrm{cok}\, g_M / \ker g_M.$$

The second condition above will be referred to as the semi-simplicity condition.

2 p-adic Height Pairings

In [8], Mazur and Tate define a p-adic height for ordinary abelian varieties, which they showed was the same as a pairing defined by Schneider in [13] (Schneider's analytic height). In the case of good ordinary reduction, Schneider shows in [15] that his analytic height is equivalent to an algebraic pairing he defines in [14]. The pairing $\langle\ ,\ \rangle_\gamma$, defined in [6] (and described below) generalizes

Schneider's algebraic height to our situation (i.e., allowing ordinary *semi-abelian* reductions). We furthermore showed in [6], that $\langle\ ,\ \rangle_\gamma$ is equivalent to the Schneider's analytic height. The pairing $\langle\ ,\ \rangle_\gamma$ is given by the following diagram.

$$
\begin{tikzcd}
\tilde{A}^0(\mathcal{O}_K)\otimes \mathbf{Z}_p \arrow[d] & \mathrm{Hom}_{\mathbf{Z}_p}(A^0(\mathcal{O}_K)\otimes \mathbf{Z}_p, \mathbf{Z}_p) \\
H^2(\mathcal{O}_K, A_{p^\infty})^* \arrow[d, "\varphi_2^*"] & (H^1(\mathcal{O}_K, A^0_{p^\infty})^*)_{\mathrm{Tor}} \arrow[u] \\
H^2(\mathcal{O}_K, A^0_{p^\infty})^* & H^1(\mathcal{O}_K, A^0_{p^\infty})^* \arrow[u] \\
{[H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})/H^1_T(\mathcal{O}_L/\mathcal{O}_K)]^*} \arrow[u, "\theta^*"] \arrow[d, "\mu^*"] & \mathcal{S}^* \arrow[u, "i^*"] \\
{[H^1(\mathcal{O}_L, A^0_{p^\infty})_\Gamma/H^1_T(\mathcal{O}_L)]^*} \arrow[d, equal] & \\
H^1(\Gamma, \mathcal{S}_\infty)^* \arrow[r, "g^*"] & H^0(\Gamma, \mathcal{S}_\infty)^* \arrow[uu, "\alpha^*"]
\end{tikzcd}
\tag{2}
$$

Explanations: We will use the notation $H^1_T(\mathcal{O}_L/\mathcal{O}_K)$ and $H^1_T(\mathcal{O}_L)$ to denote the image of the group $H^1_T(\mathcal{O}_L, A^0_{p^\infty})_\Gamma$ in $H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})$ and $H^1(\mathcal{O}_L, A^0_{p^\infty})_\Gamma$ respectively.

We define the groups

$$\mathcal{S} := \text{Image } H^1(\mathcal{O}_K, A^0_{p^\infty}) \to H^1(\mathcal{O}_K - T, A^0_{p^\infty})$$

and

$$\mathcal{S}_\infty := \text{Image } H^1(\mathcal{O}_L, A^0_{p^\infty}) \to H^1(\mathcal{O}_L - T, A^0_{p^\infty}).$$

They are quasi-isomorphic to the classical p-Selmer group of A over K and L respectively (see [6]).

The first vertical arrows in each column are obtained from the short exact sequence

$$0 \to A^0(\mathcal{O}_K)\otimes \mathbf{Z}_p \to \varprojlim H^1(\mathcal{O}_K, A_{p^\nu}) \to T_p(H^1(\mathcal{O}_K, A)) \to 0.$$

We get this by taking the inverse limit over ν in the Kummer sequence

$$0 \to \bar{A}(\mathcal{O}_K)/p^\nu A(\mathcal{O}_K) \to H^1(\mathcal{O}_K, A_{p^\nu}) \to H^1(\mathcal{O}_K, A)_{p^\nu} \to 0$$

and noting that

$$\bar{A}(\mathcal{O}_K)\otimes \mathbf{Z}_p = A^0(\mathcal{O}_K)\otimes \mathbf{Z}_p$$

(since $A^0(\mathcal{O}_K) \hookrightarrow \bar{A}(\mathcal{O}_K)$ with a finite cokernel whose order is prime to p). Since we assume that the p-primary component of $Ш_K$ is finite, $H^1(\mathcal{O}_K, A)$ is finite and so its Tate module is 0. Therefore,

$$\varprojlim H^1(\mathcal{O}_K, A_{p^\nu}) = A^0(\mathcal{O}_K)\otimes \mathbf{Z}_p.$$

We can then identify

$$\mathrm{Hom}_{\mathbf{Z}_p}(\varprojlim H^1(\mathcal{O}_K, A_{p^\nu}), \mathbf{Z}_p) \quad \text{with} \quad (H^1(\mathcal{O}_K, A^0_{p^\infty})^*)_{\mathrm{Tor}}$$

from Schneider ([15], p. 334). The fact that

$$\varprojlim H^1(\mathcal{O}_K, \tilde{A}_{p^\nu}) = H^2(\mathcal{O}_K, A_{p^\infty})^*$$

comes from McCallum's global duality ([10], Theorem 5.3)

$$H^i(\mathcal{O}_K, \tilde{A}_{p^\nu}) = H^{3-i}(\mathcal{O}_K, A_{p^\nu})^*.$$

The map φ_2^* is a special case of the maps

$$\varphi_i\colon H^i(\mathcal{O}_K, A^0_{p^\infty}) \to H^i(\mathcal{O}_K, A_{p^\infty})$$

which are quasi-isomorphisms.

The map $H^1(\mathcal{O}_K, A^0_{p^\infty})^* \to (H^1(\mathcal{O}_K, A^0_{p^\infty})^*)_{Tor}$ is the natural projection, and is a quasi-isomorphism because $H^1(\mathcal{O}_K, A^0_{p^\infty})^*$ is a finitely generated $\mathbf{Z}_p$-module, and hence, its torsion is finite.

The map i comes from the identification of $\mathcal{S}$ with the image of $H^1(\mathcal{O}_K, A^0_{p^\infty})$ in $H^1(\mathcal{O}_K - T, A^0_{p^\infty})$. Thus, i is surjective, and its kernel is bounded by the order of $H^1_T(\mathcal{O}_K, A^0_{p^\infty})$. From the exact sequence

$$H^0_T(\mathcal{O}_K, A_{p^\infty}) \to H^0_T(\mathcal{O}_K, \Phi_{p^\infty}) \to H^1_T(\mathcal{O}_K, A^0_{p^\infty}) \to H^1_T(\mathcal{O}_K, A_{p^\infty})$$

and the fact that $H^i_T(\mathcal{O}_K, A_{p^\infty}) = 0$ for $i = 0, 1$ ($i = 0$ follows from the definition and $i = 1$ is contained in the proof of Lemma 3.1 of [6]), we see that $H^1_T(\mathcal{O}_K, A^0_{p^\infty}) = H^0_T(\mathcal{O}_K, \Phi_{p^\infty}) = \Phi_{p^\infty}(\mathcal{O}_K)$ which is finite. Therefore, i is a quasi-isomorphism.

The map α is a quasi-isomorphism from Proposition 3.7 of [6].

The map θ comes from composing $H^2(\mathcal{O}_K, A^0_{p^\infty}) \to H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})$ with the projection to $H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})/H^1_T(\mathcal{O}_L/\mathcal{O}_K)$. It is a quasi-isomorphism by Proposition 7.6 of [6].

We define the map

$$\mu\colon H^1(\mathcal{O}_L, A^0_{p^\infty})_\Gamma / H^1_T(\mathcal{O}_L)_\Gamma \to H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})/H^1_T(\mathcal{O}_L/\mathcal{O}_K)$$

from the following diagram.

$$\begin{array}{ccccccccc} & & H^1_T(\mathcal{O}_L, A^0_{p^\infty})_\Gamma & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \to & H^1(\mathcal{O}_L, A^0_{p^\infty})_\Gamma & \to & H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) & \to & H^2(\mathcal{O}_L, A^0_{p^\infty})^\Gamma & \to & 0 \end{array}$$

Clearly, μ is injective, and the obstruction to it being surjective is $H^2(\mathcal{O}_L, A^0_{p^\infty})^\Gamma$.

The last equality in the first column comes from

$$H^1_T(\mathcal{O}_L, A^0_{p^\infty}) \to H^1(\mathcal{O}_L, A^0_{p^\infty}) \twoheadrightarrow \mathcal{S}_\infty.$$

Taking cohomology of Γ and use that $\mathrm{cd}_p(\Gamma) = 1$, we get the desired equality.

The map g is induced by the inclusion of $H^0(\Gamma, \mathcal{S}_\infty)$ in $\mathcal{S}_\infty$, followed by the projection to the co-invariants $(\mathcal{S}_\infty)_\Gamma = H^1(\Gamma, \mathcal{S}_\infty)$.

This finishes the definition of the height pairing. Note that all of the maps used in the definition are quasi-isomorphisms except for possibly μ^* and g^*, but μ^* is surjective.

This definition depends on our choice of generator γ of Γ. To remedy this we set

$$\langle\ ,\ \rangle_p := \langle\ ,\ \rangle_\gamma \log \kappa(\gamma).$$

2.1 Advantages of Non-degeneracy

We deduce some important implications of $\langle\ ,\ \rangle_\gamma$ being non-degenerate.

Theorem 2.1 *The pairing $\langle\ ,\ \rangle_\gamma$ is non-degenerate if and only if μ and g are quasi-isomorphisms. In this case, we also have that*

1. *$\mathcal{S}^*_\infty$ is a finitely generated torsion Λ-module which has a semi-simple action of Γ on its 1-eigenspace;*
2. *$H^2(\mathcal{O}_L, A_{p^\infty})^* = \varprojlim \tilde{A}_{p^\infty}(K_n)$;*
3. *$H^2(\mathcal{O}_L, A^0_{p^\infty})^\Gamma$ is finite of order $(\varprojlim \tilde{A}_{p^\infty}(K_n))_\Gamma$ and $H^2(\mathcal{O}_L, A^0_{p^\infty})_\Gamma = 0$.*

Proof: All but statements 2 and 3 are clear from the general nature of Λ-modules. For those, we imitate Schneider in [15], Proposition 8. We break our proof into a series of Lemmas.

Lemma 2.2 *If $\langle\ ,\ \rangle_\gamma$ is non-degenerate, $H^2(\mathcal{O}_L, A^0_{p^\infty})^\Gamma$ is finite and $H^2(\mathcal{O}_L, A^0_{p^\infty})^*$ is a torsion Λ-module.*

Proof: This follows since μ is a quasi-isomorphism and its cokernel is isomorphic to $H^2(\mathcal{O}_L, A^0_{p^\infty})^\Gamma$. The second part then follows from the structure of Λ-modules. □

Lemma 2.3 *If $\langle\ ,\ \rangle_\gamma$ is non-degenerate, $H^2(\mathcal{O}_L, A^0_{p^\infty})_{\Gamma_n} = 0$ for all n.*

Proof: From the exact sequence

$$H^2_T(\mathcal{O}_L, A^0_{p^\infty}) \to H^2(\mathcal{O}_L, A^0_{p^\infty}) \to H^2(\mathcal{O}_L - T, A^0_{p^\infty}),$$

it suffices to show that the coinvariants of each of the other terms is 0. We know that $H^2_T(\mathcal{O}_L, A^0_{p^\infty})_{\Gamma_n} = 0$ from Lemma 3.4 of [6]. Furthermore, from the Hochshild-Serre spectral sequence, we have that

$$0 \to H^2(\mathcal{O}_L - T, A^0_{p^\infty})_{\Gamma_n} \to H^3(\mathcal{O}_{K_n} - T, A^0_{p^\infty}).$$

But, $\mathcal{O}_{K_n} - T$ has p-cohomological dimension 2 (see [15], Lemma 5), and so both of these groups must be 0. Thus, $H^2(\mathcal{O}_L, A^0_{p^\infty})_{\Gamma_n} = 0$. $\square$

Therefore, $H^2(\mathcal{O}_L, A^0_{p^\infty})$ has no non-trivial finite Γ-quotients.

Lemma 2.4 *If $\langle\ ,\ \rangle_\gamma$ is non-degenerate, then there is a short exact sequence*

$$0 \to G \to H^2(\mathcal{O}_L, A^0_{p^\infty}) \to H^2(\mathcal{O}_L, A_{p^\infty}) \to 0$$

where G is a finite Γ-module.

Proof: The long exact sequence in cohomology from

$$0 \to A^0_{p^\infty} \to A_{p^\infty} \to \Phi_{p^\infty} \to 0 \tag{3}$$

gives

$$H^1(\mathcal{O}_L, \Phi_{p^\infty}) \to H^2(\mathcal{O}_L, A^0_{p^\infty}) \to H^2(\mathcal{O}_L, A_{p^\infty}) \to 0.$$

The final zero follows since

$$H^2(\mathcal{O}_L, \Phi_{p^\infty}) = \varinjlim H^2(\mathcal{O}_{K_n}, \Phi_{p^\infty}) = \varinjlim \bigoplus_v H^2(\mathcal{O}_{K_{n,v}},\ {}_v\Phi_{p^\infty})$$

where the direct sum is over the places of $\mathcal{O}_{K_{n,v}}$ where Φ_{p^∞} has support. By [7] (Lemma 5.1), these final groups can be replaced by $H^2(k(n,v),\ {}_v\Phi_{p^\infty})$ ($k(n,v)$ is the residue field of $\mathcal{O}_{K_{n,v}}$ at v) which vanishes since the absolute Galois group of the residue field, $\hat{\mathbf{Z}}$, has p-cohomological dimension 1.

The group $H^1(\mathcal{O}_L, \Phi_{p^\infty})$ has a subgroup of finite index isomorphic to $(\mathbf{Q}_p/\mathbf{Z}_p)^e$ with trivial Γ-action. The image in $H^2(\mathcal{O}_L, A^0_{p^\infty})$ then must be finite since $H^2(\mathcal{O}_L, A^0_{p^\infty})^\Gamma$ is finite. Therefore, we get the result. $\square$

By combining the previous lemma with Lemma 3.2 of [6] we see that $H^2(\mathcal{O}_{K_n}, A_{p^\infty}) \to H^2(\mathcal{O}_L, A_{p^\infty})^{\Gamma_n}$ is quasi-surjective for all n.

Lemma 2.5 *If $\langle\ ,\ \rangle_\gamma$ is non-degenerate, then $\tilde{A}_{p^\infty}(K_n) = H^2(\mathcal{O}_{K_n}, A_{p^\infty})^*$.*

Proof: This is essentially Proposition 8 of [15]. Schneider's assumption that $H^2(\mathcal{O}_{K_n}, A_{p^\infty}) \to H^2(\mathcal{O}_L/\mathcal{O}_{K_n}, A_{p^\infty})$ is surjective can be replaced by the weaker statement that $H^2(\mathcal{O}_{K_n}, A_{p^\infty}) \to H^2(\mathcal{O}_L, A_{p^\infty})^{\Gamma_n}$ is quasi-surjective, which was shown above. With these minor changes, the proof carries over to show that there is an exact sequence of Λ-modules

$$0 \to \tilde{A}_{p^\infty}(K_n) \to H^2(\mathcal{O}_{K_n}, A_{p^\infty})^* \to X \to 0$$

where X is a submodule of finite index in a free Λ-module. Since $H^2(\mathcal{O}_{K_n}, A_{p^\infty})$ is torsion (by Lemmas 2.2 and 2.4), $X = 0$. $\square$

Now to finish the proof of Theorem 2.1, take cohomology of Γ with coefficients in the exact sequence of Lemma 2.4, we get

$$0 \to G^\Gamma \to H^2(\mathcal{O}_L, A^0_{p^\infty})^\Gamma \to H^2(\mathcal{O}_L, A_{p^\infty})^\Gamma \to G_\Gamma \to 0$$

appling Lemma 2.3 for the final 0. Since G is finite, $\# G^\Gamma = \# G_\Gamma$ so

$$\# H^2(\mathcal{O}_L, A^0_{p^\infty})^\Gamma = \# H^2(\mathcal{O}_L, A_{p^\infty})^\Gamma = \#(\varprojlim \tilde{A}_{p^\infty}(K_n)^*)^\Gamma.$$

But, this is the same as $\#(\varprojlim \tilde{A}_{p^\infty}(K_n))_\Gamma$. □

2.2 A Simplification

In the case that $\langle\ ,\ \rangle_\gamma$ is non-degenerate, we can apply the results of the previous section to simplify the algebraic definition of the height pairing. The simplification will facilitate the calculation of the first non-vanishing coefficient of he p-adic L-function in Section 2.3. We start with the following lemma.

Lemma 2.6 $H^2(\mathcal{O}_L - T, A_{p^\infty}) = H^2(\mathcal{O}_L - T, A^0_{p^\infty}) = 0$.

Proof: We will first show that $H^2(\mathcal{O}_L - T, A_{p^\infty}) = 0$. By applying global duality, one sees that $H^3(\mathcal{O}_{K_n}, A_{p^\infty}) = (\varprojlim H^0(\mathcal{O}_{K_n}, A_{p^\nu}))^* = 0$ since $A_{p^\infty}(K_n)$ is finite. Hence,

$$\begin{aligned}
H^2(\mathcal{O}_L - T, A_{p^\infty}) &= \varinjlim H^2(\mathcal{O}_{K_n} - T, A_{p^\infty}) \\
&= \varinjlim \operatorname{cok}\, [H^2_T(\mathcal{O}_{K_n}, A_{p^\infty}) \to H^2(\mathcal{O}_{K_n}, A_{p^\infty})] \\
&= \varinjlim \varinjlim_{\nu} \operatorname{cok}\, [H^2_T(\mathcal{O}_{K_n}, A_{p^\nu}) \to H^2(\mathcal{O}_{K_n}, A_{p^\nu})] \\
&= (\varprojlim \varprojlim_{\nu} \ker\, [H^1(\mathcal{O}_{K_n}, \tilde{A}_{p^\nu}) \to \bigoplus_{t \in T} H^1(\mathcal{O}_{K_{n,t}}, \tilde{A}_{p^\nu})])^*
\end{aligned}$$

by McCallum's duality Theorems (4.14 and 5.3 of [10]). We know that

$$\varinjlim H^2(\mathcal{O}_{K_n}, A_{p^\infty}) = (\varprojlim \varprojlim H^1(\mathcal{O}_{K_n}, \tilde{A}_{p^\nu}))^* = (\varprojlim \tilde{A}_{p^\infty}(K_n))^*$$

by Lemma 2.5. Therefore, we need only show that the image of $\tilde{A}_{p^\nu}(K_n)$ in $H^1(\mathcal{O}_{K_n}, \tilde{A}_{p^\nu})$ maps injectively to $H^1(\mathcal{O}_{K_{n,t}}, \tilde{A}_{p^\nu})$ for all $t \in T$. But this is clear since the map on the image in question factors injectively through the image of $\tilde{A}_{p^\nu}(K_{n,t})$ in $H^1(\mathcal{O}_{K_{n,t}}, \tilde{A}_{p^\nu})$.

For $H^2(\mathcal{O}_L - T, A^0_{p^\infty})$, we use the long exact sequence coming from sequence 3 to get

$$H^1(\mathcal{O}_L - T, \Phi_{p^\infty}) \to H^2(\mathcal{O}_L - T, A^0_{p^\infty}) \to H^2(\mathcal{O}_L - T, A_{p^\infty}).$$

Since $\mathcal{O}_L - T/\mathcal{O}_K - T$ is unramified and the places of bad reduction split finitely in L, we see that $H^1(\mathcal{O}_L - T, \Phi_{p^\infty})$ is finite. Therefore $H^2(\mathcal{O}_L - T, A^0_{p^\infty})$ is also finite. But it is a quotient of $H^2(\mathcal{O}_L, A^0_{p^\infty})$, which cannot have any non-trivial finite quotients (as Γ-modules) by the remark after Lemma 2.3. Hence, $H^2(\mathcal{O}_L - T, A^0_{p^\infty})$ must be trivial. □

In our simplification, we define the map β as follows. Let

$$\mathcal{K}_2 := \ker\, H^2(\mathcal{O}_K, A^0_{p^\infty}) \to H^2(\mathcal{O}_L, A^0_{p^\infty}).$$

It is quasi-isomorphic to $H^2(\mathcal{O}_K, A^0_{p^\infty})$ since $H^2(\mathcal{O}_K, A^0_{p^\infty})^\Gamma$ is finite when the pairing is non-degenerate. The diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}_2 \arrow[r] & H^2(\mathcal{O}_K, A^0_{p^\infty}) \arrow[d] & \\
0 \arrow[r] & H^1(\mathcal{O}_L, A^0_{p^\infty})_\Gamma \arrow[r] & H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) \arrow[r] & H^2(\mathcal{O}_L, A^0_{p^\infty})^\Gamma
\end{tikzcd}$$

induces a map $\mathcal{K}_2 \to H^1(\mathcal{O}_L, A^0_{p^\infty})_\Gamma$. If we then compose with the map $H^1(\mathcal{O}_L, A^0_{p^\infty})_\Gamma \to (\mathcal{S}_\infty)_\Gamma$, we get β. It is easy to see that

$$\begin{tikzcd}
\mathcal{K}_2 \arrow[r] \arrow[d] & H^2(\mathcal{O}_K, A^0_{p^\infty}) \arrow[d, "a"] \\
H^1(\mathcal{O}_L, A^0_{p^\infty})_\Gamma / H^1_T(\mathcal{O}_L) \arrow[r, "\mu"] \arrow[d] & H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})/H^1_T(\mathcal{O}_L/\mathcal{O}_K) \\
(\mathcal{S}_\infty)_\Gamma &
\end{tikzcd}$$

commutes, and so β is quasi-isomorphism (if the height pairing is non-degenerate). Moreover, the height pairing can then be written as follows

$$\begin{tikzcd}
\tilde{A}^0(\mathcal{O}_K) \otimes \mathbf{Z}_p \arrow[d] & \mathrm{Hom}_{\mathbf{Z}_p}(A^0(\mathcal{O}_K) \otimes \mathbf{Z}_p, \mathbf{Z}_p) \\
H^2(\mathcal{O}_K, A_{p^\infty})^* \arrow[d, "\varphi_2^*"] & (H^1(\mathcal{O}_K, A^0_{p^\infty})^*)_{\mathrm{Tor}} \arrow[u] \\
H^2(\mathcal{O}_K, A^0_{p^\infty})^* \arrow[d] & H^1(\mathcal{O}_K, A^0_{p^\infty})^* \arrow[u] \\
\mathcal{K}_2^* & \mathcal{S}^* \arrow[u, "i^*"] \\
H^1(\Gamma, \mathcal{S}_\infty)^* \arrow[u, "\beta^*"] \arrow[r, "g^*"] & H^0(\Gamma, \mathcal{S}_\infty)^* \arrow[u, "\delta^*"]
\end{tikzcd} \tag{4}$$

2.3 The Calculation

If the pairing $\langle\ ,\ \rangle_\gamma$ is non-degenerate, then it can be defined by diagram 4, and we have the following result.

Theorem 2.7

$$\frac{\ker\, g}{\mathrm{cok}\, g} \approx \frac{\mathcal{L}(A)}{(\log_p \kappa(\gamma))^{r+e}} \, \frac{\det\langle\ ,\ \rangle_p \, \text{Ш}_K}{A^{fin}_{p^\infty}(K)\, \tilde{A}^{fin}_{p^\infty}(K)} \prod_{l \not| \infty} m_l.$$

Here, m_l denotes the number of connected components in the fiber of A over l (i.e., $m_l = \#\ {}_l\Phi$). The rank of $A(K)$ is denoted by r. We write

$$A_{p^\infty}(L) = A_{p^\infty}^{inf}(L) \oplus A_{p^\infty}^{fin}(L)$$

where $A_{p^\infty}^{inf}(L)$ is the divisible subgroup (this is a direct sum as Γ-modules). We then define $A_{p^\infty}^{fin}(K)$ to be the K rational points of $A_{p^\infty}^{fin}(L)$ (and similarly for $A_{p^\infty}^{inf}(K)$). Finally, we define $\mathcal{L}(A)$.

Definition 2.8 *The local $\mathcal{L}_v$-invariant of an abelian variety A with respect to the Γ-extension L/K at a place v of K which is ramified in L is*

$$\mathcal{L}_v(A) := \frac{A(K_v)/NA(K_v)}{\Phi(\mathcal{O}_{K_v})\ \ |\log_p \kappa(\gamma)|_p^{e_v}}.$$

We then have the global $\mathcal{L}$-invariant:

Definition 2.9 *The $\mathcal{L}$-invariant of A with respect to the extension L/K is*

$$\mathcal{L}(A) := \prod \mathcal{L}_v(A)$$

where the product is taken over the places v of K which ramify in L.

Remark: In many cases, one can express the local $\mathcal{L}$-invariant in terms of more basic invariants of the abelian variety. For example,

- if A has good ordinary reduction at v, then

  $$\mathcal{L}_v(A) = A(k_v)^2$$

  where k_v is the residue field of $\mathcal{O}_{K_v}$ at v, see [15];

- if A is an elliptic curve with non-split multiplicative reduction at v, then

  $$\mathcal{L}_v(A) \approx 1,$$

  see [11], (recall that $a \approx b$ iff a and b have the same p-norm);

- if A is an elliptic curve with split multiplicative reduction at v, then

  $$\mathcal{L}_v(A) = \frac{\Gamma/(q_v; L/K)\Gamma}{\text{ord } q_v\ |\log_p \kappa(\gamma)|_p} \approx \frac{\log_p q_v}{\text{ord } q_v}$$

  where q_v is the Tate parameter of A at v, and $(q_v; L/K)$ is the norm residue symbol for q_v and the extension L/K. The first equality comes from [11]. The second follows from Lemma 2.10 below.

See [5] for more examples. Note that $\mathcal{L}(A)$ is invariant under isogeny.

Lemma 2.10 *Let L_v, K_v, q_v, $\log_p$, κ, and γ be as above. Then,*

$$(q_v; L_v/K_v) \approx \frac{\log_p q_v}{\log_p \kappa(\gamma)}.$$

Proof: We have the commutative diagram

$$\begin{array}{ccc} K_v^* & \xrightarrow{\text{rec}} & \Gamma \\ \Big\downarrow{\scriptstyle \log_p} & & \Big\downarrow{\scriptstyle \kappa} \\ \mathbf{Z}_p & \xleftarrow{\log_p} & \mathbf{Z}_p^* \end{array}$$

Since we have a generator, γ, of Γ, we will consider $(q_v; L_v/K_v)$ as an element of $\mathbf{Z}_p$ and write

$$\text{rec}(q_v) \in \Gamma \quad \text{as} \quad \gamma^{(q_v; L_v/K_v)}.$$

The representation of $(q_v; L_v/K_v)$ as an element of $\mathbf{Z}_p$ depends on our choice of generator γ. A different choice would result in $(q_v; L_v/K_v)$ being represented by an element of $\mathbf{Z}_p$ which differs by a factor from $\mathbf{Z}_p^*$. Thus, the above diagram tells us that

$$\begin{aligned} \log_p q_v &\approx \log_p \kappa(\gamma^{(q_v; L_v/K_v)}) \\ &\approx \log_p [\kappa(\gamma)^{(q_v; L_v/K_v)}] \\ &\approx (q_v; L_v/K_v) \log_p \kappa(\gamma). \end{aligned}$$

Lemma 2.10 then follows. □

For the sake of simplicity, we will give the proof of Theorem 2.7 in the case where $A_{p^\infty}(L)$ is finite. We will describe at the end how this assumption can be dropped. Note that with this new assumption, $H^2(\mathcal{O}_L, A^0_{p^\infty}) = 0$ by Theorem 2.1.

We will use the following commutative diagram.

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ & & \mathcal{S} & \xrightarrow{\delta} & \mathcal{S}_\infty^\Gamma \\ & & \downarrow & & \downarrow \\ A^0_{p^\infty}(\mathcal{O}_L - T)_\Gamma & \hookrightarrow & H^1(\mathcal{O}_K - T, A^0_{p^\infty}) & \twoheadrightarrow & H^1(\mathcal{O}_L - T, A^0_{p^\infty})^\Gamma \\ & & \downarrow & & \downarrow \\ E_T & \hookrightarrow & H^2_T(\mathcal{O}_K, A^0_{p^\infty}) & \xrightarrow{\eta} & H^2_T(\mathcal{O}_L, A^0_{p^\infty})^\Gamma \\ & & \downarrow & & \downarrow \\ & & H^2(\mathcal{O}_K, A^0_{p^\infty}) & \xrightarrow{\beta} & H^1(\Gamma, \mathcal{S}_\infty) \\ & & \downarrow & & \downarrow \\ & & H^2(\mathcal{O}_K - T, A^0_{p^\infty}) & = & H^1(\Gamma, H^1(\mathcal{O}_L - T, A^0_{p^\infty})) \\ & & \downarrow & & \downarrow \\ & & 0 & & 0 \end{array} \tag{5}$$

Explanations: The first column comes from the relative cohomology sequence. The bottom 0 follows from $H^3_T(\mathcal{O}_K, A^0_{p^\infty}) = 0$.

The second column comes from taking group cohomology of the short exact sequence

$$0 \to \mathcal{S}_\infty \to H^1(\mathcal{O}_L - T, A^0_{p^\infty}) \to H^2_T(\mathcal{O}_L, A^0_{p^\infty}) \to 0$$

for the group Γ and applying Lemma 3.4 of [6] ($H^1(\Gamma, H^2_T(\mathcal{O}_L, A^0_{p^\infty})) = 0$). This sequence is exact because $H^2(\mathcal{O}_L, A^0_{p^\infty}) = 0$ by our assumption of the finiteness of $A_{p^\infty}(L)$.

The second and fifth horizontal rows come from the Hochshild-Serre spectral sequence applied to the cover $\mathcal{O}_L - T/\mathcal{O}_K - T$

$$H^p(\Gamma, H^q(\mathcal{O}_L - T, A^0_{p^\infty})) \Rightarrow H^{p+q}(\mathcal{O}_K - T, A^0_{p^\infty}).$$

It degenerates at its E_2 term because $\mathrm{cd}_p(\Gamma) = 1$. We furthermore use that $H^2(\mathcal{O}_L - T, A^0_{p^\infty}) = 0$ (see Lemma 2.6).

In the third row, we define E_T to be the kernel of η. The map η is surjective by Lemma 3.2 of [6].

Since $A_{p^\infty}(L)$ is finite, we get that $\mathcal{K}_2 = H^2(\mathcal{O}_K, A^0_{p^\infty})$. The only place where commutativity may be in question are for the boxes involving β. In these cases, one can factor η and the bottom equality through the corresponding equivariant cohomology groups. Then, commutativity follows from breaking down the natural maps between the spectral sequences

$$\begin{array}{ccc}
H^p_T(\mathcal{O}_L, R^q\pi_\Gamma A^0_{p^\infty}) & \Rightarrow & H^{p+q}_T(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) \\
\downarrow & & \downarrow \\
H^p(\mathcal{O}_L, R^q\pi_\Gamma A^0_{p^\infty}) & \Rightarrow & H^{p+q}(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) \\
\downarrow & & \downarrow \\
H^p(\mathcal{O}_L - T, R^q\pi_\Gamma A^0_{p^\infty}) & \Rightarrow & H^{p+q}(\mathcal{O}_L - T/\mathcal{O}_K - T, A^0_{p^\infty})
\end{array}$$

and

$$\begin{array}{ccc}
H^p(\Gamma, H^q_T(\mathcal{O}_L, A^0_{p^\infty})) & \Rightarrow & H^{p+q}_T(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) \\
\downarrow & & \downarrow \\
H^p(\Gamma, H^q(\mathcal{O}_L, A^0_{p^\infty})) & \Rightarrow & H^{p+q}(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) \\
\downarrow & & \downarrow \\
H^p(\Gamma, H^q(\mathcal{O}_L - T, A^0_{p^\infty})) & \Rightarrow & H^{p+q}(\mathcal{O}_L - T/\mathcal{O}_K - T, A^0_{p^\infty})
\end{array}$$

To prove Theorem 2.7 we need to look at the kernel and cokernel of g. But, since g is used to define our height pairing, we can relate part of the kernel with the determinant of our pairing.

Lemma 2.11

$|\det\langle\ ,\ \rangle_p|_p^{-1} \approx |\det\langle\ ,\ \rangle_\gamma|_p^{-1}(\log_p \kappa(\gamma))^r$

$$\approx \frac{(\ker \mathrm{Div}\, i)\ (\ker \mathrm{Div}\, \delta)\ (\ker \mathrm{Div}\, g)\ (\ker \mathrm{Div}\, \varphi_2)}{(\ker \mathrm{Div}\, \beta)\ [\tilde{A}(K)_{\mathrm{Tor}} : \tilde{A}^0(\mathcal{O}_K)_{\mathrm{Tor}}]^2}(\log_p \kappa(\gamma))^r.$$

Proof: The first equality is obvious. The second follows from a diagram chase in diagram 4. □

We can relate most of the terms in the above lemma to ker g/cok g.

Lemma 2.12

$$\frac{\ker\, g}{\mathrm{cok}\, g} \approx \mathcal{L}(A) \frac{|\det\langle\ ,\ \rangle_p|_p^{-1}}{(\log_p \kappa(\gamma))^{r+e} \ker \mathrm{Div}\, \varphi_2} \frac{\prod H^i(\mathcal{O}_K, A^0_{p^\infty})^{(-1)^{i+1}}_{\mathrm{Div}}}{[\tilde{A}(K)_{\mathrm{Tor}} : \tilde{A}^0(\mathcal{O}_K)_{\mathrm{Tor}}]^2}.$$

Proof: We rewrite the above lemma and combine it with the following calculations. A diagram chase in diagram 5 shows that β is onto and that

$$\frac{(\ker\, \beta)\ (\mathrm{cok}\, \delta)}{\ker\, \delta} = \frac{E_T}{(A^0_{p^\infty}(\mathcal{O}_L - T)_\Gamma)}$$

But, $\# A^0_{p^\infty}(\mathcal{O}_L - T)_\Gamma$ can be further simplified. Since we assume that $A_{p^\infty}(L)$ is finite, so is $A^0_{p^\infty}(\mathcal{O}_L - T)$. Thus,

$$\begin{aligned} \# A^0_{p^\infty}(\mathcal{O}_L - T)_\Gamma &= \# A^0_{p^\infty}(\mathcal{O}_L - T)^\Gamma \\ &= \# A^0_{p^\infty}(\mathcal{O}_K - T) \\ &= \# H^0(\mathcal{O}_K, A^0_{p^\infty})\ [A^0_{p^\infty}(\mathcal{O}_K - T) : A^0_{p^\infty}(\mathcal{O}_K)]. \\ &= \# H^0(\mathcal{O}_K, A^0_{p^\infty})_{\mathrm{Div}}\ [A^0_{p^\infty}(\mathcal{O}_K - T) : A^0_{p^\infty}(\mathcal{O}_K)]. \end{aligned}$$

We have calculated the order of E_T to be

$$\begin{aligned} E_T &= \prod_{t \in T} (\tilde{A}(K_t)/N\tilde{A}(K_t))_{p^\infty} \\ &\approx \prod_{t\in T} \frac{\mathcal{L}_t(\tilde{A})}{(\log_p \kappa(\gamma))^{e_t}}\ \tilde{\Phi}(\mathcal{O}_{K_t}) \end{aligned}$$

(see Lemma 3.2 of [6]).

Note that i is surjective. We compute ker i as follows:

$$\begin{aligned} \ker\, i &= \ker\, [H^1(\mathcal{O}_K, A^0_{p^\infty}) \to \mathcal{S}] \\ &= \frac{H^1_T(\mathcal{O}_K, A^0_{p^\infty})}{\mathrm{cok}\, [A^0_{p^\infty}(\mathcal{O}_K) \to A^0_{p^\infty}(\mathcal{O}_K - T)]} \\ &= \frac{\prod_{t\in T} H^1_!(\mathcal{O}_{K,t}, A^0_{p^\infty})}{[A^0_{p^\infty}(\mathcal{O}_K - T) : A^0_{p^\infty}(\mathcal{O}_K)]} \\ &= \frac{\prod_{t\in T} \Phi(\mathcal{O}_{K_t})_{p^\infty}}{[A^0_{p^\infty}(\mathcal{O}_K - T) : A^0_{p^\infty}(\mathcal{O}_K)]}. \end{aligned}$$

Note that the number of components for A and $\tilde{A}$ are the same, and that $\mathcal{L}(A) = \mathcal{L}(\tilde{A})$. Putting the above information together, we get the lemma. □

To simplify the Euler characteristic, we relate it to one calculated by Schneider.

Lemma 2.13

$$\frac{\prod H^i(\mathcal{O}_K, A^0_{p^\infty})^{(-1)^{i+1}}_{\mathrm{Div}}}{\ker\ \mathrm{Div}\,\varphi_2} = \frac{\prod H^i(\mathcal{O}_K, A_{p^\infty})^{(-1)^{i+1}}_{\mathrm{Div}}}{\ker\ \mathrm{Div}\,\varphi_1}.$$

Proof: We take cohomology over $\mathcal{O}_K$ of

$$0 \to A^0_{p^\infty} \to A_{p^\infty} \to \Phi_{p^\infty} \to 0$$

and use that $\Phi_{p^\infty}(\mathcal{O}_K)$ and $H^1(\mathcal{O}_K, \Phi_{p^\infty})$ have the same order. □

Schneider (in [14]) calculates the Euler product in terms of some classical invariants of the abelian variety. The lemma is stated in the case where A has good ordinary reduction at the primes ramifying in L, but the proof carries over in our situation (including the related Lemmas 6.7(i) and 8.3). One only has to substitute McCallum's global duality theorem ([10], Theorem 5.3) for Mazur/Artin duality used by Schneider. Also, the assumption that $H^1(\mathcal{O}_K, A)_{p^\infty}$ is finite is satisfied by our assumption that $Ш_{p^\infty}$ is finite (see Section 1).

Lemma 2.14 (Schneider)

$$\prod_i H^i(\mathcal{O}_K, A_{p^\infty})^{(-1)^{i+1}}_{\mathrm{Div}} \approx \frac{Ш_K}{A_{p^\infty}(K)\ \tilde{A}_{p^\infty}(K)} \prod_{l \not| \infty} m_l\ |[\tilde{A}(K)_{\mathrm{Tor}} : \tilde{A}^0(\mathcal{O}_K)_{\mathrm{Tor}}]|_p.$$

The final error term can be handled as follows.

Lemma 2.15

$$\ker\ \mathrm{Div}\,\varphi_1 = |[\tilde{A}(K)_{\mathrm{Tor}} : \tilde{A}^0(\mathcal{O}_K)_{\mathrm{Tor}}]|_p^{-1}.$$

Proof: We can see this once we note that

$$\begin{aligned} \mathrm{Div}\, H^1(\mathcal{O}_K, A_{p^\infty}) &= A(\mathcal{O}_K) \otimes \mathbf{Q}_p/\mathbf{Z}_p, \\ \mathrm{Div}\, H^1(\mathcal{O}_K, A^0_{p^\infty}) &= A^0(\mathcal{O}_K) \otimes \mathbf{Q}_p/\mathbf{Z}_p, \end{aligned}$$

and take the appropriate definitions into account. □

From the above lemmas we see that

$$\frac{\ker\ g}{\mathrm{cok}\ g} = \frac{\mathcal{L}(A)}{(\log_p \kappa(\gamma))^{r+e}}\ \frac{\det\langle\ ,\ \rangle_p\ Ш_K}{A_{p^\infty}(K)\ \tilde{A}_{p^\infty}(K)} \prod_{l \not| \infty} m_l.$$

This, of course, is the same as the formula in the statement of Theorem 2.7 when $A_{p^\infty}(L)$ is finite. If $A_{p^\infty}(L)$ is infinite, then we no longer have diagram 5 and have to break it into a series of several smaller diagrams. In this case, the maps $\mathcal{K}_2 \to H^2(\mathcal{O}_K, A^0_{p^\infty})$ and β are no longer surjective, and quotient of the orders of their cokernels is $\#H^2(\mathcal{O}_L, A^0_{p^\infty})^\Gamma = \#\tilde{A}^{inf}_{p^\infty}(K)$. Also, one now has that

$$\#A^0_{p^\infty}(\mathcal{O}_L - T)_\Gamma = \frac{\#H^0(\mathcal{O}_K, A^0_{p^\infty})_{\mathrm{Div}}}{A^{inf}_{p^\infty}(K)}\,[A^0_{p^\infty}(\mathcal{O}_K - T) : A^0_{p^\infty}(\mathcal{O}_K)]$$

instead of the value calculated above.

3 Choice of Λ-Module

In this section we exhibit several Λ-modules which could equally serve for defining the Iwasawa L-function of a semi-stable ordinary abelian variety. Each alternative reduces to the classical Selmer group (up to quasi-isomorphism) in the case of good ordinary reduction.

The results of the previous section indicate that any module H such that one has a quasi-exact sequence

$$0 \to \mathcal{S}_\infty \to H \to (\mathbf{Q}_p/\mathbf{Z}_p)^e \to 0$$

or

$$0 \to (\mathbf{Q}_p/\mathbf{Z}_p)^e \to H \to \mathcal{S}_\infty \to 0$$

would suffice provided that Γ acted trivially on the $(\mathbf{Q}_p/\mathbf{Z}_p)^e$ term. In either case one still gets the equation

$$f_H(t) = (t-1)^e f_{\mathcal{S}}(t).$$

We define our candidate for the p-adic L-function of an ordinary abelian variety which is semi-stable at p as follows. Let

$$L_H(s) := f_H(\kappa(\gamma)^{1-s}).$$

We let $\rho =$ the order of vanishing of $L_H(s)$ at 1, $r =$ rank of $A(K)$ over $\mathbf{Z}$, and state our main theorem.

Theorem 3.1

$$\rho \geq r + e.$$

If $\langle\ ,\ \rangle_p$ is non-degenerate, then

$$\rho = r + e$$

and

$$L_H^{(\rho)}(1) \approx \mathcal{L}(A)\,\frac{\det\langle\ ,\ \rangle_p\ \text{Ш}_K}{A^{fin}_{p^\infty}(K)\ \tilde{A}^{fin}_{p^\infty}(K)}\prod_{l \not| \infty} m_l.$$

Proof: Under our hypotheses,

$$L_H(s) = F_H((\kappa(\gamma))^{1-s} - 1) = (\kappa(\gamma)^{1-s} - 1)^e F_{\mathcal{S}}((\kappa(\gamma))^{1-s} - 1).$$

The first statement is obvious. If the pairing is non-degenerate, then g is a quasi-isomorphism, and we put together Theorem 2.7, the fact that

$$f_{\mathcal{S}}^{(r)}(1) = \frac{\ker\, g}{\mathrm{cok}\, g},$$

and

$$L_H^{(\rho)}(1) = (-\log_p \kappa(\gamma))^\rho f_{\mathcal{S}}^{(r)}(1).$$

The fact that $\rho = r + e$ is clear in this case. □

If $A_{/\mathbf{Q}}$ is an elliptic curve over $\mathbf{Q}$, we can use the calculations of the $\mathcal{L}$-invariant made in Section 2.3 to give the formula a form which matches the value predicted by the conjectures in [9]. (In this case, $A_{p^\infty}(L)$ is automatically be finite, so $A_{p^\infty}(\mathbf{Q}) = A_{p^\infty}^{fin}(\mathbf{Q})$.) If A is the Néron model of an elliptic curve with non-split multiplicative reduction at p, then

$$\rho \geq r.$$

Furthermore, if $\langle\ ,\ \rangle_p$ is non-degenerate, then

$$\rho = r$$

and

$$L_H^{(\rho)}(1) \approx \frac{\det\langle\ ,\ \rangle_p\ \text{Ш}_K}{A_{p^\infty}(K)\ \tilde{A}_{p^\infty}(K)} \prod_{l \not| \infty} m_l.$$

If A is the Néron model of an elliptic curve with split multiplicative reduction at p, then

$$\rho \geq r + 1.$$

Furthermore, if $\langle\ ,\ \rangle_p$ is non-degenerate, then

$$\rho = r + 1$$

and

$$L_H^{(\rho)}(1) \approx \frac{\log_p q}{\mathrm{ord}\ q}\ \frac{\det\langle\ ,\ \rangle_p\ \text{Ш}_K}{A_{p^\infty}(K)\ \tilde{A}_{p^\infty}(K)} \prod_{l \not| \infty} m_l.$$

where q is the Tate parameter of A at p.

3.1 $H^1(\mathcal{O}_L, A^0_{p^\infty})$

The first such module we will look at is $H^1(\mathcal{O}_L, A^0_{p^\infty})$. On one hand, we will see that it satisfies the expected semi-simplicity condition. Its major drawback is that it cannot be easily described in terms of the Galois module $A_{p^\infty}(\overline{K})$ as one would hope.

Lemma 3.2 *If the Néron model of A over $\mathcal{O}_K$ is semi-stable at all primes dividing p, then there is a short exact sequence of Λ-modules*

$$0 \to H \to H^1(\mathcal{O}_L, A^0_{p^\infty}) \to \mathcal{S}_\infty \to 0,$$

where H is isomorphic as an abelian group to the direct sum of $(\mathbf{Q}_p/\mathbf{Z}_p)^e$ with a finite group. Furthermore, there is a quasi-exact sequence

$$0 \to (\mathbf{Q}_p/\mathbf{Z}_p)^e \to H^1(\mathcal{O}_L, A^0_{p^\infty})^\Gamma \to \mathcal{S}^\Gamma_\infty \to 0.$$

Proof: The first short exact sequence comes from the long exact sequence in relative cohomology over $\mathcal{O}_L$

$$A^0_{p^\infty}(\mathcal{O}_L) \to A^0_{p^\infty}(\mathcal{O}_L - T) \to H^1_T(\mathcal{O}_L, A^0_{p^\infty}) \to H^1(\mathcal{O}_L, A^0_{p^\infty}) \to H^1(\mathcal{O}_L - T, A^0_{p^\infty})$$

First we note that

$$\begin{aligned} H^1_T(\mathcal{O}_L, A^0_{p^\infty}) &= \bigoplus_{t\in T} H^1_!(\mathcal{O}_{L_t}, A^0_{p^\infty}) \\ &= \bigoplus_{t\in T} \varinjlim H^1_!(\mathcal{O}_{K_{n,t}}, A^0_{p^\infty}) \\ &= \bigoplus_{t\in T} \varinjlim H^1_!(\mathcal{O}_{K_{n,t}}, A^0)_{p^\infty} \quad (\text{Since } H^0_!(\mathcal{O}_{K_{n,t}}, A^0) = 0) \\ &= \bigoplus_{t\in T} \varinjlim \Phi(\mathcal{O}_{K_{n,t}})_{p^\infty} \end{aligned}$$

From the p-adic uniformization of the abelian variety ([3]), we see that this final group has a subgroup of finite index isomorphic to $(\mathbf{Q}_p/\mathbf{Z}_p)^e$. Since $\mathbf{Q}_p/\mathbf{Z}_p$ is a divisible group,

$$H = \bigoplus_{t\in T} \varinjlim \Phi(\mathcal{O}_{K_{n,t}})_{p^\infty} = (\mathbf{Q}_p/\mathbf{Z}_p)^e \oplus F$$

as abelian groups (where F is finite).

So, by putting $\mathcal{S}_\infty$ in the above sequences, all we need is the finiteness of $A^0_{p^\infty}(\mathcal{O}_L - T)/A^0_{p^\infty}(\mathcal{O}_L)$, which is supplied by Lemma 3.3 below. Thus, we get the first short exact sequence.

For the quasi-exact sequence, we note that Γ acts trivially on the $(\mathbf{Q}_p/\mathbf{Z}_p)^e$, to get the first term. To get the quasi-surjection, note that up to quasi-isomorphism,

$$\mathcal{S}_\infty^\Gamma \sim \mathcal{S} \sim H^1(\mathcal{O}_K, A^0_{p^\infty}).$$

Since $H^1(\mathcal{O}_K, A^0_{p^\infty})$ maps to $H^1(\mathcal{O}_L, A^0_{p^\infty})^\Gamma$, the sequence quasi-splits, which implies that the map has to be quasi-surjective. $\square$

Lemma 3.3 *The group $A_{p^\infty}(\mathcal{O}_L)/A^0_{p^\infty}(\mathcal{O}_L)$ is finite.*

Proof: From the long exact sequence in cohomology

$$0 \to A^0_{p^\infty}(\mathcal{O}_L) \to A_{p^\infty}(L) \to \Phi_{p^\infty}(\mathcal{O}_L)$$

we see that the assertion is equivalent to showing that the points of $A_{p^\infty}(L)$ lie on only a finite number of components of $\Phi(\mathcal{O}_L)$. Thus, it suffices to show that the map $A_{p^\infty}(L_t) \to \Phi(\mathcal{O}_{L_t})$ has finite image for each $t \in T$. Furthermore, we can make a finite unramified extension to reduce to the case where the torus in the reduction of A at t is split. Then, $Gal(L_t/K_t)$ acts trivially on $\Phi(\mathcal{O}_{L_t})$.

We let $\gamma \in Gal(L_t/K_t)$ denote a topological generator and look at the homomorphism

$$\gamma - 1 \colon A_{p^\infty}(L_t) \to A_{p^\infty}(L_t).$$

Since γ acts trivially on Φ, the image of $\gamma - 1$ is contained in $A^0_{p^\infty}(\mathcal{O}_{L_t})$. So, if $A_{p^\infty}(L_t)/A^0_{p^\infty}(\mathcal{O}_{L_t})$ is infinite, $\gamma - 1$ has an infinite cokernel and hence, an infinite kernel. But, the kernel of $\gamma - 1$ is $A_{p^\infty}(K_t)$, which is finite. $\square$

Proposition 3.4 *There is a quasi-exact sequence*

$$0 \to H^1(\mathcal{O}_K, A^0_{p^\infty}) \to H^1(\mathcal{O}_L, A^0_{p^\infty})^\Gamma \to (\mathbf{Q}_p/\mathbf{Z}_p)^e \to 0.$$

Proof: This follows easily from the preceding Lemma and the following exact commutative diagram.

$$\begin{array}{ccccccc}
 & & & & H^1(\mathcal{O}_K, A^0_{p^\infty}) & \to & \mathcal{S} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & H^\Gamma \oplus (\mathbf{Q}_p/\mathbf{Z}_p)^e & \to & H^1(\mathcal{O}_L, A^0_{p^\infty})^\Gamma & \to & \mathcal{S}_\infty^\Gamma & &
\end{array}$$

If we require only quasi-exactness in the diagram, the lemma above gives

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & H^1(\mathcal{O}_K, A^0_{p^\infty}) & \to & \mathcal{S} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & (\mathbf{Q}_p/\mathbf{Z}_p)^e & \to & H^1(\mathcal{O}_L, A^0_{p^\infty})^\Gamma & \to & \mathcal{S}_\infty^\Gamma & \to & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

The proposition is clear, from a diagram chase. $\square$

From these Propositions, we see that $H^1(\mathcal{O}_L, A^0_{p^\infty})$ is quasi-isomorphic to $\mathcal{S}_\infty \oplus (\mathbf{Q}_p/\mathbf{Z}_p)^e$. Therefore, its dual is semi-simple on its 1-eigenspace.

In this situation, one can consider the p-adic Zeta-function of A. We let $F_i(T)$ be the characteristic polynomial of the Λ-module $H^i(\mathcal{O}_L, A^0_{p^\infty})$. Then, the Zeta-function is defined by

$$Z_{(L/K;A)}(s) := \prod_i F_i(\kappa(\gamma)^{1-s} - 1)^{(-1)^{i+1}}.$$

If $A_{p^\infty}(L)$ is finite, then this reduces immediately to the L-function described above. If not, then there are contributions made in dimensions 0 and 2 which do not change the order of vanishing but do change the first non-vanishing coefficient to be the expected expression

$$\mathcal{L}(A)\,\frac{\det\langle\ ,\ \rangle_p\ \mathit{III}_K}{A_{p^\infty}(K)\ \tilde{A}_{p^\infty}(K)}\ \prod_{l\not|\infty} m_l.$$

Note that if we chose the equivariant group $H^1(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})$ to be our analogue of the p-Selmer group over K for A, then $H^1(\mathcal{O}_L/\mathcal{O}_L, A^0_{p^\infty}) = H^1(\mathcal{O}_L, A^0_{p^\infty})$ would still be our group over L. However, we have the short exact sequence from the descent sequence in equivariant cohomology (see appendix to [14])

$$0 \to H^0(\mathcal{O}_L, A^0_{p^\infty})_\Gamma \to H^1(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) \to H^1(\mathcal{O}_L, A^0_{p^\infty})^\Gamma \to 0.$$

The first group is finite, and we have seen that the last group is quasi-isomorphic to $(\mathbf{Q}_p/\mathbf{Z}_p)^{r+e}$ (assuming that the p-primary component of III_K is finite). In other words, this group exhibits the extra zeros over K (a cohomological analogue of Mazur, Tate and Teitelbaum's extended Mordell-Weil group [9]).

3.2 $H^1(\mathcal{O}_L, A_{p^\infty})$

We now consider the module $H^1(\mathcal{O}_L, A_{p^\infty})$. It has the appeal that it is a subgroup of the Galois cohomology group $H^1(\ /L, A_{p^\infty}(\overline{L}))$. However, we will show below that it does not always satisfy the semi-simplicity condition.

From the exact sequence 3,

$$0 \to A^0_{p^\infty} \to A_{p^\infty} \to \Phi_{p^\infty} \to 0,$$

we take the long exact sequence in cohomology over $\mathcal{O}_L$. Since $H^2(\mathcal{O}_L, A^0_{p^\infty})$ is finite, the following sequence is exact with finite cokernel

$$0 \to \mathbf{S}_{clas}(L) \to H^1(\mathcal{O}_L, A_{p^\infty}) \to H^1(\mathcal{O}_L, \Phi_{p^\infty}).$$

But $H^1(\mathcal{O}_L, \Phi_{p^\infty})$ is quasi-isomorphic to $(\mathbf{Q}_p/\mathbf{Z}_p)^e$ (with a trivial action of Γ) and $\mathbf{S}_{clas}(L)$ is quasi-isomorphic to $\mathcal{S}_\infty$. So $H^1(\mathcal{O}_L, A_{p^\infty})$ is a suitable module for defining the L-function.

To show that its dual may not be semi-simple on the 1-eigenspace for γ, we specialize to the case where $K = \mathbf{Q}$ (so $\mathcal{O}_K = \mathbf{Z}$, and L must be the cyclotomic Γ-extension), and A is an elliptic curve with split multiplicative reduction at p.

If the height pairing is non-degenerate, then we know that $H^1(\mathcal{O}_L, A^0_{p^\infty})$ is a torsion Λ-module, and consequently, $H^1(\mathcal{O}_L, A_{p^\infty})$ is also. The order of vanishing of the characteristic polynomial of $H^1(\mathcal{O}_L, A^0_{p^\infty})^*$ at 0 is the same as that of $H^1(\mathcal{O}_L, A_{p^\infty})^*$, both of which are equal to the order of vanishing at 0 of the polynomial for $\mathcal{S}^*_\infty$ plus 1. By Proposition 1.1, $H^1(\mathcal{O}_L, A_{p^\infty})$ would satisfy the semi-simplicity condition if the $\mathbf{Z}_p$-rank of $(H^1(\mathcal{O}_L, A_{p^\infty})^\Gamma)^*$ is equal to the order of vanishing of the characteristic polynomial of $H^1(\mathcal{O}_L, A_{p^\infty})^*$ at zero. But this is the $\mathbf{Z}_p$-rank of $H^1(\mathbf{Z}, A_{p^\infty})^*$ plus one. The non-semi-simplicity of $H^1(\mathcal{O}_L, A_{p^\infty})$, when the rank of A over $\mathbf{Q}$ is greater than 1, is then shown by the following proposition.

Proposition 3.5 *Let A be an elliptic curve over $\mathbf{Q}$ which has rank ≥ 1 with split multiplicative reduction mod p. Then, the natural map*

$$H^1(\mathbf{Z}, A_{p^\infty}) \to H^1(\mathcal{O}_L, A_{p^\infty})^\Gamma$$

is a quasi-isomorphism.

Proof: We begin with the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & A_{p^\infty}(L)_\Gamma & & E_p & & \\
 & & \downarrow & & \downarrow & & \\
H^1(\mathbf{Z}, A_{p^\infty}) & \hookrightarrow & H^1(\mathbf{Z}-p, A_{p^\infty}) & \to & H^2_p(\mathbf{Z}, A_{p^\infty}) & \to & H^2(\mathbf{Z}, A_{p^\infty}) \\
\downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow & & \\
H^1(\mathcal{O}_L, A_{p^\infty})^\Gamma & \hookrightarrow & H^1(\mathcal{O}_L - p, A_{p^\infty})^\Gamma & \to & H^2_p(\mathcal{O}_L, A_{p^\infty})^\Gamma & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

By a diagram chase, the kernel of α is contained in $A_{p^\infty}(L)_\Gamma$ which we already know is finite. The cokernel of α is a sub-quotient of $H^2_p(\mathbf{Z}, A_{p^\infty})$. We can compute this group as follows. From the long exact sequence in cohomology of short exact sequence (see Section 1)

$$0 \to A_{p^\nu} \to A \to \bar{A} \to 0$$

and taking the direct limit over ν we get

$$0 \to H^1_{\cdot}(\mathbf{Z}_p, \bar{A}) \otimes \mathbf{Q}_p/\mathbf{Z}_p \to H^2_{\cdot}(\mathbf{Z}_p, A_{p^\infty}) \to H^2_{\cdot}(\mathbf{Z}_p, A)_{p^\infty}.$$

Since $H^{1}_{?}(\mathbf{Z}_p, \bar{A})$ is a torsion group, its tensor product with $\mathbf{Q}_p/\mathbf{Z}_p$ is zero. Therefore, we have an injection

$$H^{2}_{?}(\mathbf{Z}_p, A_{p^\infty}) \to H^{2}_{?}(\mathbf{Z}_p, A)_{p^\infty}.$$

But, $H^{2}_{?}(\mathbf{Z}_p, A)$ is dual to $A^0(\mathbf{Z}_p)$. Since A is split multiplicative at p, the Tate parameterization tells us that

$$A^0(\mathbf{Z}_p) \cong \mathbf{Z}_p^* = \mathbf{Z}/(p-1)\mathbf{Z} \oplus \mathbf{Z}_p.$$

Thus, $H^{2}_{?}(\mathbf{Z}_p, A)_{p^\infty} = \mathbf{Q}_p/\mathbf{Z}_p$, and E_p is a subgroup of it. Therefore, either it is finite, or it is isomorphic to $\mathbf{Q}_p/\mathbf{Z}_p$. If it is finite we are done, so we will consider the latter case.

The cokernel of α is isomorphic to the elements of E_p which vanish when mapped to $H^2(\mathbf{Z}, A_{p^\infty})$ modulo the image of $A(\mathcal{O}_L)_\Gamma$. Therefore, we will be done when we show that the map $H^{2}_{?}(\mathbf{Z}_p, A_{p^\infty}) \to H^2(\mathbf{Z}, A_{p^\infty})$ has a finite kernel. We denote this kernel by ker.

By duality, we have

$$\begin{aligned} ker &= \varinjlim \ker\, [H^{2}_{?}(\mathbf{Z}_p, A_{p^\nu}) \to H^2(\mathbf{Z}, A_{p^\nu})] \\ &= (\varprojlim \operatorname{cok}\, [H^1(\mathbf{Z}, A_{p^\nu}) \to H^1(\mathbf{Z}_p, A_{p^\nu})])^* \end{aligned}$$

We determine this cokernel by using Kummer sequences:

$$\begin{array}{ccccccc} 0 & \to & \bar{A}(\mathbf{Z})/p^\nu A(\mathbf{Z}) & \to & H^1(\mathbf{Z}, A_{p^\nu}) & \to & H^1(\mathbf{Z}, A)_{p^\nu} \\ & & \downarrow f_1 & & \downarrow f_2 & & \downarrow \\ 0 & \to & \bar{A}(\mathbf{Z}_p)/p^\nu A(\mathbf{Z}_p) & \to & H^1(\mathbf{Z}_p, A_{p^\nu}) & \to & H^1(\mathbf{Z}_p, A) \end{array}$$

We know from [10], Proposition 3.1 that $H^1(\mathbf{Z}_p, A)$ is finite (its order is equal to the number of components in the fiber over p in the Néron model of A over $\mathbf{Z}_p$). We also know that $A^0(\mathbf{Z}_p) \cong \mathbf{Z}_p^*$, and therefore,

$$\bar{A}(\mathbf{Z}_p)/p^\nu A(\mathbf{Z}_p) = \mathbf{Z}/p^n\mathbf{Z}$$

where $\nu - n$ does not depend on ν. If the rank of A is greater or equal to 1, then the vertical map f_1 is surjective, because it is induced by the inclusion of $\mathbf{Z}^{rank}$ into $\mathbf{Z}_p$. Thus, the cokernel of f_2 is finite, and bounded independently of ν. Therefore, ker is finite. □

3.3 Greenberg's Module

Here we will only mention that Ralph Greenberg has defined a Selmer group in [4]. He defines it in terms of Galois cohomology for a certain class of p-adic representations. Thus, it has the advantage of being defined in greater generality than the modules described above. If the module has an extra zero, Greenberg defines in terms of the module what should correspond to the "$\log q$" factor in a conjectured formula for the first non-vanishing coefficient. He, furthermore, shows that his module satisfies one of the properties above if the representation comes from the natural Galois action on p-power torsion points on an elliptic curve with split multiplicative reduction. So, the corresponding L-function has the desired order of vanishing and first non-vanishing coefficient by Proposition 3.1. This Selmer group, however, is in general not semi-simple. The module resembles $H^1(\mathcal{O}_L, A_{p^\infty})$ a great deal, and may in fact be quasi-isomorphic to it.

3.4 The Functional Equation for $\mathcal{S}$

We derive functional equation for $f_{\mathcal{S}}(t)$. We then get functional equations for any of the $f_H(t)$ from the previous section.

Proposition 3.6

$$f_{\mathcal{S}}(t) = (-1)^\epsilon t^\lambda f_{\mathcal{S}}(1/t)$$

where λ is the λ-invariant of $\mathcal{S}$ and ϵ multiplicity of the root 1 in $f_{\mathcal{S}}(t)$.

Proof: Mazur's result [7] reduces us to establishing a self pairing on $\mathcal{S}_n/\mathrm{Div}$ (i.e., $\mathcal{S}_n$ modulo its divisible subgroup) such that the kernels of the pairing are torsion groups killed by an integer m chosen independently of n. We note that $\mathcal{S}_n/\mathrm{Div}$ is quasi-isomorphic to the image of $H^1(\mathcal{O}_{K_n}, A^0_{p^\infty})/\mathrm{Div}$ in $H^1(\mathcal{O}_{K_n}, A_{p^\infty})/\mathrm{Div}$, and that the order of the kernel and the cokernel of the map are bounded independently of n. Then, from the Kummer theory of A, we have the exact sequences

$$0 \to A^0(\mathcal{O}_{K_n}) \otimes \mathbf{Q}_p/\mathbf{Z}_p \to H^1(\mathcal{O}_{K_n}, A^0_{p^\infty}) \to H^1(\mathcal{O}_{K_n}, A^0)_{p^\infty} \to 0$$

and

$$0 \to \bar{A}(\mathcal{O}_{K_n}) \otimes \mathbf{Q}_p/\mathbf{Z}_p \to H^1(\mathcal{O}_{K_n}, A_{p^\infty}) \to H^1(\mathcal{O}_{K_n}, A)_{p^\infty}.$$

The second sequence does not necessarily map surjectively onto $H^1(\mathcal{O}_{K_n}, A)_{p^\infty}$, but the cokernel's order is finite and bounded independently of n. Since the left hand terms in the two sequences are divisible, we can make the identifications of

$$H^1(\mathcal{O}_{K_n}, A^0_{p^\infty})/\mathrm{Div} \quad \text{with} \quad H^1(\mathcal{O}_{K_n}, A^0)_{p^\infty}/\mathrm{Div}$$

and

$$H^1(\mathcal{O}_{K_n}, A_{p^\infty})/\mathrm{Div} \quad \text{with} \quad H^1(\mathcal{O}_{K_n}, A)_{p^\infty}/\mathrm{Div}.$$

Accordingly, $\mathcal{S}_n/\mathrm{Div}$ is quasi-isomorphic to the image of

$$H^1(\mathcal{O}_{K_n}, A^0)_{p^\infty}/\mathrm{Div} \quad \text{in} \quad H^1(\mathcal{O}_{K_n}, A)_{p^\infty}/\mathrm{Div}.$$

By the global duality for Néron models, we know that

$$\varprojlim_{m} H^1(\mathcal{O}_{K_n}, A^0)/m \quad \text{is dual to} \quad H^1(\mathcal{O}_{K_n}, \tilde{A})/\mathrm{Div}\,.$$

On p-parts we get

$$\varprojlim_{\nu} H^1(\mathcal{O}_{K_n}, A^0)/p^\nu \quad \text{is dual to} \quad H^1(\mathcal{O}_{K_n}, \tilde{A})_{p^\infty}/\mathrm{Div}\,.$$

But, since multiplication by p^ν is surjective on the divisible part of a group, and $H^1(\mathcal{O}_{K_n}, A^0)$ is a torsion group, we have that

$$\begin{aligned}\varprojlim_{\nu} H^1(\mathcal{O}_{K_n}, A^0)/p^\nu &= \varprojlim_{\nu}(H^1(\mathcal{O}_{K_n}, A^0)/\mathrm{Div}\,)/p^\nu \\ &= (H^1(\mathcal{O}_{K_n}, A^0)/\mathrm{Div}\,)_{p^\infty} \\ &= (H^1(\mathcal{O}_{K_n}, A^0)_{p^\infty})/\mathrm{Div}\,.\end{aligned}$$

The last two equalities come from the fact that we are dealing now with finite p-groups. So, we find that

$$H^1(\mathcal{O}_{K_n}, A^0)_{p^\infty}/\mathrm{Div} \quad \text{is dual to} \quad H^1(\mathcal{O}_{K_n}, \tilde{A})_{p^\infty}/\mathrm{Div}\,.$$

Similarly, we get that

$$H^1(\mathcal{O}_{K_n}, A)_{p^\infty}/\mathrm{Div} \quad \text{is dual to} \quad H^1(\mathcal{O}_{K_n}, \tilde{A}^0)_{p^\infty}/\mathrm{Div}\,.$$

For the Selmer groups, we then see that

$$\mathcal{S}_n/\mathrm{Div} \quad \text{is almost dual to} \quad \tilde{\mathcal{S}_n}/\mathrm{Div}$$

where the kernels of the duality pairing are bounded independently of n. Since, A and $\tilde{A}$ are isogenous, there are quasi-isomorphisms between

$$H^i(\mathcal{O}_{K_n}, A) \quad \text{and} \quad H^i(\mathcal{O}_{K_n}, \tilde{A})$$

and

$$H^i(\mathcal{O}_{K_n}, A^0) \quad \text{and} \quad H^i(\mathcal{O}_{K_n}, \tilde{A}^0).$$

Therefore, $\mathcal{S}_n/\mathrm{Div}$ is quasi-isomorphic to $\tilde{\mathcal{S}_n}/\mathrm{Div}$, and we have a self-duality on $\mathcal{S}_n/\mathrm{Div}$ which has the required property. □

From the equation

$$f_H(t) = (t-1)^e f_S(t).$$

we get the following Corollary.

Corollary 3.7

$$f_H(t) = (-1)^\rho t^\lambda f_H(1/t)$$

where λ is the λ-invariant of H, and ρ is the multiplicity of 1 as a root of $f_H(t)$.

Similarly, we have a functional equation for $L_H(s)$:

$$L_H(s) = (-1)^\rho \kappa(\gamma)^{\lambda(1-s)} L_H(2-s)$$

References

[1] Bernardi, D., Goldstein, C.: Variante p-adique de la conjecture de Birch et Swinnerton-Dyer. C. R. Acad. Sci. Paris **301**, 455-458 (1985)

[2] Bernardi, D., Goldstein, C., Stephens, N.: Notes p-adiques sur les courbes elliptiques. Reine Angew. Math. **351**, 129-170 (1984)

[3] Bosch, S., Lütkebohmert, W.: Stable reduction and uniformization of abelian varieties II. Invent. math. **78**, 257-298 (1984)

[4] Greenberg, R.: Iwasawa theory for p-adic representations. (To appear)

[5] Jones, J.: Galois cohomology and parameterizations of semi-stable abelian varieties. (In preparation)

[6] Jones, J.: p-adic heights for semi-stable Abelian varieties. (submitted)

[7] Mazur, B.: Rational points on abelian varieties with values in towers of number fields. Invent. math. **18**, 183-266 (1972)

[8] Mazur, B., Tate, J.: "Canonical heights via biextensions", in Arithmetic and Geometry, vol I, Birkäuser, 195-238 (1983)

[9] Mazur, B., Tate, J., Teitelbaum, J.: On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer. Invent. math. **84** 1-48 (1986)

[10] McCallum, W.: Duality theorems for Neŕon models. Duke Math. J. **53** 1093-1124 (1986)

[11] Nasybullin, A.: Elliptic Tate curves over local Γ-extensions. Math. Notes of U.S.S.R. **13** 322-327 (1973)

[12] Perrin-Riou, B.: Descente infinie et hauteur p-adique sur les courbes elliptiques á multiplication complexe. Invent. math. **70**, 369-398 (1982)

[13] Schneider, P.: p-adic height pairings I. Invent. math. **69**, 401-409 (1982)

[14] Schneider, P.: Iwasawa L-functions of varieties over algebraic number fields. A first approach. Invent. math. **71**, 251-293 (1983)

[15] Schneider, P.: p-adic height pairings II. Invent. math. **79**, 329-374 (1985)