

Minimal Solvable Nonic Fields

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ABSTRACT

For each solvable Galois group which appears in degree 9 and each allowable signature, we find polynomials which define the fields of minimum absolute discriminant.

1. Introduction

For a degree n extension K/\mathbf{Q} , let D_K denote its discriminant and K^{gal} its Galois closure. If we fix a positive integer n , a bound B , and a transitive subgroup $G \leq S_n$, one can try to compute all number fields K with $[K : \mathbf{Q}] = n$, $\text{Gal}(K^{gal}/\mathbf{Q}) = G$, and $|D_K| \leq B$. If G contains an element of order 2 whose cycle type consists of r_2 transpositions and r_1 one-cycles, one can ask the same question for fields with signature $[r_1, r_2]$.

For solvable transitive subgroups of S_n , minimal fields of this sort are completely known through degree 8 (see [KM01] for results and references). In degree $n = 9$, the only results in the literature are from [Voi08] and [DyDO95]. In [Voi08], Voight computes all totally real nonic fields with bounded discriminant using Hunter's theorem and Martinet's generalization, and has subsequently extended these computations [Voi12]. In [DyDO95], Diaz y Diaz and Olivier consider the first question for each signature for imprimitive fields, those with non-trivial subfields, and also employ Martinet's generalization of Hunter's theorem. Taken together, these results did not find minimum fields for many combinations of group/signature.

Here, we determine the fields with minimal discriminant for the all solvable nonic Galois groups for each allowable signature, using Galois theory and class field theory. Table 1.1 gives an overview, showing the minimal root discriminant for each group/signature. Of the 34 Galois groups in degree 9, 30 of them are solvable, the exceptions being $\text{PSL}_2(8)$, $\text{PTL}(2, 8)$, A_9 , and S_9 . We note that of the nonsolvable groups, little is known about minimal discriminants. Takeuchi [Tak99] determined the totally real S_9 field with smallest discriminant, which has root discriminant ≈ 12.87 , and [Voi12] has the first 21 such fields. For other signatures/groups, the minimums are not known, but the web site for [KM01] has S_9 fields with root discriminant 6.76, 7.82, 9.16, and 10.74 for $r_1 = 1, 3, 5$, and 7 respectively.

The computation involves both standard techniques and applications of the tame-wild principle, a new approach to obtaining relations among discriminants of subfields of a Galois number field. It has the advantage of being easy to apply, and the relations it finds are optimal.

In Section 2 we set some basic notation and give a background on methods used in more than one section. Most cases work from base fields of degree 6 or 8, and in some cases we had to compute these fields as well. Section 2.3 describes additional techniques we used to compute base fields. Section 3 treats imprimitive nonics: Section 3.1 covers the easiest case where the nonic field has non-trivial automorphisms and Section 3.2 treats imprimitive nonic fields with only the trivial automorphism. Section 4 treats primitive solvable nonic fields, which is the most difficult case. This is the primary place we used the tame-wild principle, which is described in Section 4.2.

TABLE 1.1. Overview of minimal root discriminants. Combinations which are not possible are grayed out. Galois groups are given by their T -numbers.

$G \setminus r_1$	1	3	5	7	9
1					13.70
2					15.83
3	9.72				22.74
4		8.38			13.75
5	14.29				57.56
6					17.63
7					26.09
8	8.93	11.51			24.63
9	19.92				49.76
10	9.57				27.97
11	14.67				51.00
12		8.92			21.94
13		10.98			30.32
14	21.52				80.16
15	17.74				62.74
16	9.19	10.91			21.84
17					14.93
18	8.53	10.12			19.53
19	18.80	17.89			43.57
20		7.83			13.65
21	9.82				27.97
22		10.27			25.96
23	16.48				64.85
24	9.15	12.46			29.14
25			12.89		26.27
26	17.47	12.79			34.51
28		8.18	9.19	11.15	13.63
29		9.38	13.47		22.98
30	6.86		13.18		25.81
31	6.83	8.08	10.30	12.95	15.55

2. Notation

As mentioned above, the discriminant of an extension K/\mathbf{Q} will be denoted by D_K . We will be more concerned with its absolute discriminant, $d_K := |D_K|$. Some discriminant information is more conveniently expressed in terms of root discriminants; let $\text{rd}(K) := d_K^{1/[K:\mathbf{Q}]}$.

When finding all fields K with a given Galois group and $d_K \leq B$ for some bound B , we will phrase the question in terms of root discriminants: determine fields with $\text{rd}(K) \leq B_{rd}$ where $B_{rd} = B^{1/[K:\mathbf{Q}]}$.

We will adopt the convention that K_j will denote a degree j extension of \mathbf{Q} . We will then abbreviate its absolute discriminant, d_{K_j} , by d_j . For relative extensions L/K , the discriminant ideal will be denoted by $\mathfrak{d}_{L/K}$.

For transitive subgroups of S_n , we use the T -numbering of [BM83] and write nT_j for the j -th transitive subgroup (see [LMF12] for data on groups with this numbering). In the special case of $n = 9$, we abbreviate this to T_j . We denote the cyclic group of order n by C_n .

Let $\mathbf{Q} \subseteq K \subseteq L$ be number fields. Then, it is well-known that

$$d_K^{[L:K]} \mathbb{N}_{K/\mathbf{Q}}(\mathfrak{d}_{L/K}) = d_L$$

where $\mathbb{N}(I)$ denotes the numeric norm of an ideal I . In particular, $d_K^{[L:K]} \mid d_L$ and

$$\text{rd}(K) \leq \text{rd}(L). \quad (2.1)$$

2.1. Galois theory

Nonic fields with more than one cubic subfield are the easiest to compute; class field theory is unnecessary since such a field is the compositum of two of its subfields. The relevant nonic Galois groups are T_j for $j = 2, 4, 5$, and 8 .

If K_3 is a subfield of K_9 , then $\text{rd}(K_3) \leq \text{rd}(K_9)$, so it is easy to determine which cubic fields to consider. This may lead to computing many composita and taking their discriminants, but we can accelerate the process by pre-screening pairs of fields. If K_3 and K'_3 are a pair of candidate cubic subfields, then the contribution of a prime p to $\text{rd}(K_9)$ is greater than or equal to the maximum of its contribution to $\text{rd}(K_3)$ and $\text{rd}(K'_3)$. So, one first computes the discriminant of each candidate cubic subfield in factored form. Then for each pair, one quickly computes a lower bound for $\text{rd}(K_9)$, and only if this is $\leq B_{rd}$ do we compute the compositum, its root discriminant, and if necessary, its Galois group.

2.2. Class field theory

The class field theory functions of the program `gp` can compute defining equations for cyclic extensions of prime degree. Except in Section 3.1, the desired solvable nonic field K_9 is not an abelian extension of a subfield, so we need to work with subfields of K_9^{gal} : a base field M and a cyclic cubic extension L/M such that $L^{gal} = K_9^{gal}$. Computing K_9 from L can be done directly with Magma [BCP97], but we found it convenient in many cases to do this by constructing resolvents using `gp` [PAR10] which was used for most of the other parts of the computation.

Let G be a solvable nonic Galois group. Below, we give the Galois group of a minimum degree base field M , which will depend G . This is determined by a group theoretic computation which we completed using `gap` [GAP06]. The main question then becomes how to compute appropriate discriminant bounds for the base fields M and for their extensions L , which will depend in part on the group G .

2.3. Additional techniques used in computing base fields

Each computation relies on having a complete set of base fields with a given degree and Galois group with discriminants up to some bound. In some cases, we employed techniques beyond those discussed elsewhere in this paper to carry out these auxiliary computations.

The most difficult nonic field computations are where the nonic is primitive. Here the base field for class field theory is usually an octic field. Three types of fields which required additional work were those with Galois groups $8T_5 \cong Q_8$, $8T_8 \cong QD_{16}$, $8T_{23} \cong \text{GL}_2(3)$. The following comments apply to these three types of octic fields. Each can be computed as a quadratic extension of a quartic field. We make use of two improvements in carrying out this computation. The first comment below restricts the set of base fields we need to consider, and the second restricts the set of moduli for a given base field.

First we note that octic fields with Galois group $8T_5$, $8T_8$, and $8T_{23}$ have quartic subfields with Galois groups $4T_2$, $4T_3$, and $4T_5$ respectively. However, not every quartic field of a given type can be extended to the appropriate type of octic, and so we filter potential base fields based on lifting criteria. In the first two cases, we used criteria from [GSS95], and we used the criterion from [JR99] for passing from $4T_5 \cong S_4$ to $\text{GL}_2(3) \cong \tilde{S}_4$.

Second, the kernel of the natural map from the octic Galois group to the quartic Galois group has order 2 in each case, and is generated by an element with cycle type 2222. Thus, if a prime $p > 2$ is unramified in the quartic field but ramified in the octic field, its contribution

to the conductor of the relative extension is simply p . Not having to consider factors of p in the quartic field greatly reduces the number of conductors we have to consider.

2.4. Guide to the remaining sections

Each nonic group G treated below using class field theory involves a C_3 extension of a base field, where the base has degree 3, 4, 6, or 8 over \mathbf{Q} . For each G , we give the Galois group of the base field. For cases where we use Galois theory, we list $G_1 \otimes G_2$ where the G_i are the Galois groups of the cubic subfields as described above.

These tables describe the searches without regard to signature. They give the bound B_{rd} used for the computation for the given group, the number of nonic number fields up to isomorphism with the given Galois group which satisfy $\text{rd}(K_9) \leq B_{rd}$, and the minimum root discriminant for that group. This bound is sufficient for determining the fields with minimal discriminant for each pair (G, r_1) , with the exception of the totally real case for some groups. An additional search was then conducted for totally real fields with the only differences being larger discriminant bounds, and the restriction to totally real base fields. For the sake of space, we do not give additional details on the totally real searches. All of the polynomials found by our computations can be accessed at the web site [JR].

We note that for each group and signature, there is a unique number field with minimum discriminant with one exception: for $G = T_{30}$ and $r_1 = 5$, there are a pair of non-isomorphic fields with the same minimal discriminant.

3. Imprimitive fields

We describe the computations and results for nonic fields which contain at least one cubic subfield. We divide these fields into two types based on whether or not the nonic field has non-trivial automorphisms.

3.1. Nonic fields with automorphisms

There are 8 groups T_j where the corresponding nonic field has non-trivial automorphisms: T_j with $j = 1, 2, 4, 6, 7, 12, 17,$ and 20 .

Let K_9 be a corresponding nonic field. Since K_9/\mathbf{Q} has non-trivial automorphisms, it is a C_3 extension of a subfield K_3 . We can apply class field theory directly to candidate subfields K_3 . Since $\text{rd}(K_3) \leq \text{rd}(K_9)$, we first find all candidates K_3 with $\text{rd}(K_3) \leq B_{rd}$, and then find C_3 extensions K_9 with $\text{rd}(K_9) \leq B_{rd}$. Table 3.1 gives an overview of the computation.

TABLE 3.1. Overview of nonic fields with non-trivial automorphisms.

Group	T_1	T_2	T_4	T_6	T_7	T_{12}	T_{17}	T_{20}
Base	$3T_1$	$3T_1 \otimes 3T_1$	$3T_1 \otimes 3T_2$	$3T_1$	$3T_1$	$3T_2$	$3T_1$	$3T_2$
B_{rd}	30	30	15	30	30	15	30	15
# fields found	3	4	40	2	2	52	110	291
Min rd	13.7	15.8	8.4	17.6	26.1	8.9	14.9	7.8

3.2. Imprimitive nonic fields with trivial automorphism group

There are 15 nonic groups where the degree 9 field has only the trivial automorphism and has a cubic subfield: T_j with $j = 3, 5, 8, 10, 11, 13, 18, 21, 22, 24, 25, 28, 29, 30,$ or 31 .

Let K_9 be a corresponding nonic field, and K_3 a cubic subfield. Since K_9 has no non-trivial automorphisms, the relative extension K_9/K_3 is a cubic S_3 extension. Accordingly, in K_9^{gal} there is a quadratic extension K_6/K_3 , and then a C_3 extension K_{18}/K_6 such that K_{18} contains K_9 . We need bounds on d_6 and d_{18} .

For an S_3 extension, if we write \mathfrak{d}_j for the relative discriminant of the degree j subfield over base, there are two discriminant relations we make use of:

$$\mathfrak{d}_2 \mathfrak{d}_3^2 = \mathfrak{d}_6 \quad \text{and} \quad \mathfrak{d}_2 \mid \mathfrak{d}_3. \tag{3.1}$$

These can be proved in a variety of ways, for example by combining subfield inclusions with character relations, or by the tame-wild principle [JR12] which holds for any subgroup of S_5 with arbitrary base field.

In our case, the base is K_3 , so we can take norms down to \mathbf{Q} to yield $\mathbb{N}_{K_3/\mathbf{Q}}(\mathfrak{d}_j) = d_{3j}/d_3^j$. The divisibility condition from equation (3.1) gives

$$\frac{d_6}{d_3^2} \mid \frac{d_9}{d_3^3} \implies d_6 d_3 \mid d_9$$

and so if $d_9 \leq B$,

$$d_6 d_3 \leq B.$$

The equality from equation (3.1) gives

$$\frac{d_6}{d_3^2} \left(\frac{d_9}{d_3^3} \right)^2 = \frac{d_{18}}{d_3^6} \implies d_6 d_9^2 = d_{18} d_3^2.$$

We get the bound

$$d_{18} \leq \frac{B^2 d_6}{d_3^2}. \tag{3.2}$$

The group G determines the Galois group of the relevant sextic field K_6 . We start with sextic fields with that Galois group such that $d_6 \leq B$, then using their cubic subfields prune the list so that $d_6 d_3 \leq B$. Then, for each field K_6 , we have d_3 and d_6 and then use inequality 3.2 to compute the fields K_{18} .

Table 3.2 gives an overview of the computation. We note that both T_5 and T_6 involve the

TABLE 3.2. Overview of imprimitive nonic fields with trivial automorphism group.

Group	T_3	T_5	T_8	T_{10}	T_{11}	T_{13}	T_{18}	T_{21}	T_{22}
Base	$6T_2$	$3T_2 \otimes 3T_2$	$3T_2 \otimes 3T_2$	$6T_2$	$6T_2$	$6T_1$	$6T_3$	$6T_2$	$6T_1$
B_{rd}	15	15	21.54	15	15	21.54	12.6	15	15
# fields found	9	1	439	11	1	39	18	55	19
Min rd	9.7	14.3	8.9	9.6	14.7	11.0	8.5	9.8	10.3

Group	T_{24}	T_{25}	T_{28}	T_{29}	T_{30}	T_{31}
Base	$6T_3$	$6T_4$	$6T_6$	$6T_7$	$6T_8$	$6T_{11}$
B_{rd}	15	15	15	15	15	15
# fields found	175	6	3912	53	63	84996
Min rd	9.2	12.9	8.2	9.4	6.9	6.8

composita of two S_3 cubics. For a polynomial f , if we denote its discriminant by $\text{disc}(f)$, then T_5 fields arise from cubic S_3 polynomials f_1 and f_2 where $\mathbf{Q}(\sqrt{\text{disc}(f_1)}) = \mathbf{Q}(\sqrt{\text{disc}(f_2)})$ (equivalently, $\text{disc}(f_1) \cdot \text{disc}(f_2)$ is a square), and T_6 when the fields are not equal.

4. Primitive fields

The solvable Galois groups for nonic fields which are primitive are T_j for $j = 9, 14, 15, 16, 19, 23, \text{ or } 26$. An overview is given in Table 4.1.

Here we employed two methods for dealing with discriminant bounds. For all but $j = 16, 19, \text{ and } 26$ we were able to use results of computations of fields with small Galois root discriminant (defined below). In the case $j = 16$ this approach was sufficient to find the field with smallest absolute discriminant without regard to signature. However, to find minimal fields for all relevant signatures, we used additional discriminant relations which we discovered using the tame-wild principle.

TABLE 4.1. Overview of primitive solvable nonic fields.

Group	T_9	T_{14}	T_{15}	T_{16}	T_{19}	T_{23}	T_{26}
Base	$4T_1$	$8T_5$	$8T_1$	$8T_4$	$8T_8$	$8T_{12}$	$8T_{23}$
B_{rd}	29.33	29.33	29.33	12.6	20	21.54	20
# fields found	15	2	3	25	6	3	56
Min rd	19.9	21.5	17.7	9.2	17.9	16.5	12.8

4.1. Galois root discriminants

If K is a number field, the *Galois root discriminant* of K is the root discriminant of K^{gal} , which we denote by $\text{grd}(K) := \text{rd}(K^{gal})$. Applying equation 2.1 to $K \subseteq K^{gal}$ gives

$$\text{rd}(K) \leq \text{grd}(K). \quad (4.1)$$

If $K \subseteq L$, then equation 2.1 applied to $K^{gal} \subseteq L^{gal}$ gives

$$\text{grd}(K) \leq \text{grd}(L). \quad (4.2)$$

In [JW12], we address (with Wallington) the problem: given a finite group G and a bound B_{grd} , to find all number fields K such that $\text{Gal}(K^{gal}/\mathbf{Q}) \cong G$ and $\text{grd}(K) \leq B_{grd}$. In some ways, this grd -analog of our problem is simpler: by inequality (4.2) one has a fixed bound for all intermediate fields K_j in the calculation, namely $\text{grd}(K_j) \leq B_{grd}$. One has to deal with Galois root discriminants, but [JW12] provides means for handling this. In [JW12], the grd -analog of our problem is solved for solvable nonic Galois groups with bound $B_{grd} = 8\pi e^\gamma \approx 44.7632$, a constant introduced by Serre [Ser86]. The origins of this constant will not play a role here, but it is the asymptotic limit for root discriminant bounds assuming the Generalized Riemann Hypothesis. In some cases, we extended the computations from [JW12] here.

It remains to connect results from this grd -analog to our original problem. Inequality (4.1) is in the wrong direction to be of use. To find an inequality which points in the correct direction, we first consider tame extensions. The approach we describe has a second application here, so we start with a more general description.

4.2. The tame-wild principle

Let L/\mathbf{Q} be a finite Galois extension with Galois group G . There is a natural bijection between permutation representations $\rho: G \rightarrow S_N$ and degree N algebras over \mathbf{Q} which are isomorphic to products of subfields of L , with both sides of the correspondence taken up to isomorphism. If $\rho: G \rightarrow S_N$ is a permutation representation and L/\mathbf{Q} a Galois extension with $\text{Gal}(L/\mathbf{Q}) = G$, then we denote the associated algebra by A_ρ . If A is such an algebra, we write

d_A for its absolute discriminant, and if p is a prime, let $c_p(A)$ be the valuation of d_A with respect to p .

Suppose p is a prime, A a degree N algebra as above, $\rho : G \rightarrow S_N$ the corresponding permutation representation, and I is the inertia subgroup for a prime above p . If p is tamely ramified, then $I = \langle \sigma \rangle$. If the cycle type of $\rho(\sigma)$ is $n_1, n_2, \dots, n_{\ell(\rho(\sigma))}$ (including 1-cycles), then

$$c_p(A) = \sum_{j=1}^{\ell(\rho(\sigma))} (n_j - 1) = N - \ell(\rho(\sigma)) = \deg(\rho) - \ell(\rho(\sigma)).$$

We define the formal discriminant exponent for ρ and σ by $\mathfrak{c}_\rho(\sigma) = \deg(\rho) - \ell(\rho(\sigma))$.

To compare discriminants from two algebras, A_ρ and A_ν arising from L/K with corresponding permutation representations ρ and ν , we consider constants $\alpha \in \mathbf{Q}$ such that

$$\mathfrak{c}_\rho(\sigma) \leq \alpha \cdot \mathfrak{c}_\nu(\sigma) \quad \text{for all } \sigma \in G. \tag{4.3}$$

If inequality 4.3 holds, then for any prime p which is tamely ramified in L/\mathbf{Q} we have $c_p(A_\rho) \leq \alpha \cdot c_p(A_\nu)$.

Let G be a finite group and ρ and ν two permutation representations of G where ν is non-trivial. Let

$$\alpha(G, \rho, \nu) = \max_{\substack{\sigma \in G \\ \mathfrak{c}_\nu(\sigma) \neq 0}} \frac{\mathfrak{c}_\rho(\sigma)}{\mathfrak{c}_\nu(\sigma)}.$$

We say that the tame-wild principle holds for (G, ρ, ν) if for every Galois extension L/\mathbf{Q} with Galois group G , $d_{A_\rho} \leq d_{A_\nu}^{\alpha(G, \rho, \nu)}$. We say that it holds unconditionally for G if it holds for (G, ρ, ν) for all pairs of permutation representations ρ and ν .

Let $i : G \hookrightarrow S_N$ be an embedding of a transitive subgroup of S_N , and $r_G : G \rightarrow S_{|G|}$ the left regular representation. The following theorem asserts that the tame-wild principle always holds for (G, r_G, i) . To make the result directly applicable to our situation, it has been normalized for root discriminants, and writes the constant on the left hand side. It gives a simple formula for the constant $\alpha(G, r_G, i)$ with this normalization, which we denote by $\underline{\alpha}(G)$. Let $\mathcal{F}(G)$ be the maximum number of fixed points of a non-identity element of $i(G)$.

THEOREM 4.2.1 [JR12], Thm. 7.3. *Let K be a degree n extension of \mathbf{Q} and $G = \text{Gal}(K^{gal}/\mathbf{Q})$ viewed as a subgroup of S_n . Then,*

$$\text{grd}(K)^{\underline{\alpha}(G)} \leq \text{rd}(K)$$

where $\underline{\alpha}(G) = 1 - \frac{\mathcal{F}(G)}{n}$.

It is easy to see that in computing $\mathcal{F}(G)$, it suffices to check elements of prime order. For groups G where we used grd -bounds, Table 4.2 shows the value $\underline{\alpha}(G)$, the bound B_{grd} , and the corresponding bound B_{rd} .

TABLE 4.2. For several nonic groups given by their T -number t , the value B_{grd} for which we searched, $\underline{\alpha}(G)$, and the corresponding value B_{rd} rounded down to 2 decimal places.

t	9	14	15	16	18	23
B_{grd}	44.76	44.76	44.76	44.76	44.76	100
$\underline{\alpha}(T_t)$	8/9	8/9	8/9	2/3	2/3	2/3
B_{rd}	29.33	29.33	29.33	12.6	12.6	21.54

We now describe the three groups which required additional discriminant bounds.

4.3. T_{16}

We now give details on an application of the tame-wild principle to the computation of T_{16} fields.

We say that a finite group is *inertial* if there is a prime p such that the group is an extension of a cyclic group of order prime to p by a p -group. A group G has a *complete partition* if $G - \{e\}$ can be written as a disjoint union of sets of the form $H - \{e\}$ where each H is cyclic (see [Suz50]).

PROPOSITION 4.3.1. *The tame-wild principle holds universally for the group $T_{16} \cong C_3^2 : D_4$.*

Proof. This follows by combining results from [JR12]. In particular, to prove that the tame-wild principle holds universally for a group G , it suffices to check that it holds for all inertial subgroups of G . Moreover, it holds universally for groups with a complete partition ([JR12], Thm. 5.3). The only inertial subgroups of T_{16} which do not have a complete partition are isomorphic to $C_3 \times S_3$, and the tame-wild principle holds universally for this group as well ([JR12], Thm. 6.2). So, the tame-wild principle holds universally for T_{16} . \square

For $T_{16} \cong C_3^2 : D_4$ extensions, we are interested in relating discriminants of the degree 9 field, an octic resolvent field with Galois group $8T_4 \cong D_4$, the dihedral group of order 8, and a degree 24 field which is a C_3 extension of the octic field. For each conjugacy class of T_{16} , we computed the cycle type of a representative for its action on roots of a degree n defining polynomial for our extension with $n = 8, 9$, and 24. Table 4.3 shows the corresponding values

K_9	K_8	K_{24}
3 ₂₂₂	4 ₂₂₂₂	12 ₂₂₂₂₂₂₂₂₂₂₂₂₂₂
6 ₄₄	6 ₄₄	18 ₄₄₄₄₄₄
4 ₂₂₂₂	4 ₂₂₂₂	12 ₂₂₂₂₂₂₂₂₂₂₂₂₂₂
7 ₆₃	4 ₂₂₂₂	16 ₆₆₂₂₂₂₂₂
7 ₆₃	4 ₂₂₂₂	20 ₆₆₆₆
6 ₃₃₃	0	16 ₃₃₃₃₃₃₃₃
6 ₃₃₃	0	8 ₃₃₃₃

TABLE 4.3. Rows correspond to non-identity conjugacy classes of T_{16} . For the relevant permutation representations of degrees 9, 8, and 24, we give c_{type} where *type* is the cycle type of a representative element σ , and c is the corresponding value $\mathbf{c}_\rho(\sigma)$.

of $\mathbf{c}_\rho(\sigma)$ subscripted by the corresponding cycle type. From the table, we can read off the strongest relations which hold for all rows. Since $4/3$ times the first column is always greater than or equal to the second column, $d_8 \leq d_9^{4/3}$. Similarly, $d_{24} \leq d_9^{8/3} d_8$. The first relation is used to compute the needed octic fields, and the second to bound the discriminants of their degree 24 overfields.

4.4. T_{19} and T_{26}

For both T_{19} and T_{26} , we compute C_3 extensions $K_{24} = L$ of an octic field $K_8 = M$. We need bounds for d_8 and d_{24} .

For the group $G = T_{26}$, the degree 24 field K_{24} has Galois group $24T_{1326}$ and K_8 has Galois group $8T_{23}$. Let ψ_j denote the permutation character corresponding to the degree j subfield.

Then it is easy to check that $\psi_{24} = 2\psi_9 + \psi_8$, and so we have the discriminant relation

$$d_{24} = d_9^2 d_8. \quad (4.4)$$

Since the $8T_{23}$ field K_8 is a subfield of K_{24} , $d_8^3 \mid d_{24}$. Thus,

$$d_8^3 \mid d_9^2 d_8 \implies d_8 \mid d_9. \quad (4.5)$$

The group T_{19} is a subgroup of T_{26} , so it has corresponding resolvent algebras of degrees 8 and 24 which turn out to be fields. The character relation then carries over to the case of T_{19} . Thus, the same relations from equations (4.4) and (4.5) hold when working with T_{19} . The only relevant difference then in carrying out the computation is the Galois group for the octic field is $8T_8$.

In each case, we start with all octic fields of the desired Galois group such that $d_8 \leq B$, and compute C_3 extensions so that the degree 24 field satisfies $d_{24} \leq d_8 B^2$. For T_{19} , we carried out this computation in general to find fields with $\text{rd}(K_9) \leq 18$, and for T_{26} , so that $\text{rd}(K_9) \leq 20$, and with higher bounds in the totally real cases.

References

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