MAT 442: Test 1 Solutions

Throughout, $F$ is a field.

1. In each part, say whether the statement is true or false. Then, either give a brief reason why the statement is true, or a counterexample showing that it is false.

   (a) If $V$ is a vector space over $\mathbb{R}$ and $u, v \in V$ with $u \neq v$, then $\{u, v\}$ is a linearly independent set.
   
   This is false. For example, if $u = (1, 0)$ and $v = (2, 0)$, they satisfy the hypotheses, but $2u - v = 0$, so they are not linearly independent.

   (b) If $v_1, \ldots, v_k$ are linearly independent elements of a vector space $V$, then every basis for $V$ contains $v_1, \ldots, v_k$.
   
   This is false. There exists a basis containing the given set of vectors, but not every basis contains them. For example, $V = \mathbb{R}^2$, $v_1 = (1, 1)$, then $\{v_1\}$ is a linearly independent set of vectors. But, the basis $\{(1, 0), (0, 1)\}$ does not contain $v_1$.

   (c) If $V$ is a vector space with a basis consisting of 7 vectors, then all bases for $V$ have 7 vectors.
   
   This is true, every basis of a vector space has the same number of elements.

   (d) If $V$ is a vector space with a spanning set consisting of 7 vectors, then all spanning sets for $V$ have 7 vectors.
   
   This is false. If you have a spanning set, then you can add vectors to the set without changing the span. Here we can take $V = \mathbb{R}$, and $\{1, 2, 3, 4, 5, 6, 7\}$ is a spanning set of 7 vectors. But, $\{1\}$ is a spanning set with just one element.

   (e) The space of $2 \times 3$ matrices over $F$ is isomorphic to $P_5(F)$ (the vector space of polynomials of degree less than or equal to 5).
   
   This is true. Both vector spaces are 6 dimensional, and two vector spaces are isomorphic if and only if they have the same dimension.

2. Suppose $V$ is a vector space over $F$. Prove that for all $a \in F$,

   $$a\vec{0} = \vec{0}$$

   where $\vec{0}$ is the zero vector of $V$.

   For all $a \in F$, $a\vec{0} = a(\vec{0} + \vec{0}) = a\vec{0} + a\vec{0}$. Then subtract $a\vec{0}$ from both sides to get $\vec{0} = a\vec{0}$.

3. (a) Define the term basis.

   A basis is a subset of a vector space $V$ which is linearly independent and which spans $V$.

   (b) Suppose $V$ is a vector space, $S$ a finite spanning set of $V$. Prove that $V$ has a basis.

   The proof is in the book.

(a) **Prove that $UT$ is a linear transformation.**

For all $x, y \in V$ and all $c \in F$,

\[
UT(x + cy) = U(T(x + cy)) = U(T(x) + cT(y)) = U(T(x)) + cU(T(y)) = UT(x) + cUT(y)
\]

def’n of composition

So, $UT$ is linear.

(b) **Define null space (i.e. $N(T)$) and nullity (i.e. $\text{nullity}(T)$).**

$N(T) = \{x \in V \mid T(x) = \vec{0}\}$ and $\text{nullity}(T)$ is the dimension of $N(T)$.

(c) **Prove that $N(T) \subseteq N(UT)$.**

If $x \in N(T)$, then $T(x) = \vec{0}$. But then, $UT(x) = U(T(x)) = U(\vec{0}) = \vec{0}$, so $x \in N(UT)$.

(d) (10 points) **Prove that if $V$ and $W$ are finite dimensional, then**

**nullity$(T) \leq \text{nullity}(UT)$.**

Since $N(T)$ is a subspace of $N(UT)$, $\dim(N(T)) \leq \dim(N(UT))$ (take a basis for $N(T)$, it is a linearly independent subset of $N(UT)$ so it can be extended to a basis of $N(UT)$). But then $\text{nullity}(T) \leq \text{nullity}(UT)$.

5. Suppose $W_1$ and $W_2$ are finite dimensional subspaces of a vector space $V$.

(a) **Prove that $W_1 + W_2$ is a subspace of $V$.**

Recall $W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1$ and $w_2 \in W_2\}$. Note, $\vec{0} \in W_i$ for $i = 1, 2$, so $\vec{0} + \vec{0} \in W_1 + W_2$, so $W_1 + W_2$ is not empty.

Then for all $w_1 + w_2, w'_1 + w'_2 \in W_1 + W_2$ and $k \in F$ where $w_i, w'_i \in W_i$ for $i = 1, 2$, we have $(w_1 + w_2) + k(w'_1 + w'_2) = w_1 + k w'_1 + w_2 + k w'_2$, and each $w_i + k w'_i \in W_i$ because $W_i$ is a subspace. Therefore $W_1 + W_2$ is a subspace.

(b) **Prove that $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.**

Everything is in a finite dimensional vector space $V$, so $W_1$, $W_2$, and $W_1 \cap W_2$ are all finite dimensional. Pick a basis $v_1, \ldots , v_m$ of $W_1 \cap W_2$. Extend this to a basis $v_1, \ldots , v_m, w_1, \ldots , w_r$ of $W_1$ and also extend it to a basis $v_1, \ldots , v_m, w'_1, \ldots , w'_s$ of $W_2$. So, $\dim(W_1) = m + r, \dim(W_2) = m + s, \text{ and } \dim(W_1 \cap W_2) = m$. It will suffice to show that $v_1, \ldots , v_m, w_1, \ldots , w_r, w'_1, \ldots , w'_s$ is a basis for $W_1 + W_2$ since then $\dim(W_1 + W_2) = m + r + s = (m + r) + (m + s) - m$ as desired.
First, if \( u_1 + u_2 \in W_1 + W_2 \) with \( u_i \in W_i \) for \( i = 1, 2 \), then there exist scalars \( a_i, b_i, c_i, d_i \in F \) such that

\[
\begin{align*}
  u_1 &= \sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{r} b_i w_i \\
  u_2 &= \sum_{i=1}^{m} c_i v_i + \sum_{i=1}^{s} d_i w'_i 
\end{align*}
\]

so

\[
 u_1 + u_2 = \sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{r} b_i w_i + \sum_{i=1}^{m} c_i v_i + \sum_{i=1}^{s} d_i w'_i = \sum_{i=1}^{m} (a_i + c_i) v_i + \sum_{i=1}^{r} b_i w_i + \sum_{i=1}^{s} d_i w'_i 
\]

Thus, \( v_1, \ldots, v_m, w_1, \ldots, w_r, w'_1, \ldots, w'_s \) spans \( W_1 + W_2 \).

Suppose we have a linear combination which is zero:

\[
\vec{0} = \sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{r} b_i w_i + \sum_{i=1}^{s} d_i w'_i 
\]

Then, \( u = -\sum_{i=1}^{s} d_i w'_i = \sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{r} b_i w_i \) is in \( W_1 \cap W_2 \) (first expression is in \( W_2 \) and second is in \( W_1 \)). Therefore, it equals some linear combination of just the \( v_i \). Using that \( v_1, \ldots, v_m, w_1, \ldots, w_r \) is a basis for \( W_1 \), we get that all \( b_i = 0 \). So,

\[
\vec{0} = \sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{s} d_i w'_i
\]

but now all \( a_i = 0 \) and all \( d_i = 0 \) because \( v_1, \ldots, v_m, w'_1, \ldots, w'_s \) is a basis for \( W_2 \). Thus, our set \( v_1, \ldots, v_m, w_1, \ldots, w_r, w'_1, \ldots, w'_s \) is linearly independent, and spans \( W_1 + W_2 \), so we are done.

(c) Suppose that \( \dim(V) = 5 \) and \( \dim(W_1) = \dim(W_2) = 3 \). Prove that there exists \( w \in W_1 \cap W_2 \) with \( w \neq 0 \).

Since \( W_1 + W_2 \subseteq V \), we have \( \dim(W_1 + W_2) \leq 5 \). Applying part (b), we have

\[
5 \geq \dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = 3 + 3 - \dim(W_1 \cap W_2)
\]

This implies \( \dim(W_1 \cap W_2) \geq 1 \), so it contains a non-zero vector.