MAT 442: Test 1 Solutions

Throughout, $F$ is a field.

1. Suppose $V$ is a vector space over $F$. Prove that for all $v \in V$, 

$$0_F v = 0$$

where $0_F$ denotes the zero element of the field $F$, and $0$ denotes the zero element of $V$. 
(This is a theorem from the course.)

For all $v \in V$, 

$$0_F v = (0_F + 0_F) v = 0_F v + 0_F v.$$ 

Now add $-0_F v$ to both sides to get 

$$0 = -0_F v + 0_F v = -0_F v + (0_F v + 0_F v) = (-0_F v + 0_F v) + 0_F v = 0 + 0_F v = 0_F v$$

2. (a) State the Dimension Theorem (also known as the Rank-Nullity Theorem).

See the text.

(b) Prove this theorem. (This is a theorem from the course.)

See the text, Thm. 2.3.

3. Suppose $T: V \rightarrow W$ is a linear transformation, and $S \subseteq V$. Prove 

$$T(\text{span}(S)) = \text{span}(T(S))$$

(This was proved in the course. Prove it from the definition of span and linear transformation.)

If $S = \emptyset$, then $T(\text{span}(\emptyset)) = T(\{0\}) = \{0\}$ and $\text{span}(T(\emptyset)) = \text{span}(\emptyset) = \{0\}$. So, $T(\text{span}(S)) = \text{span}(T(S))$. Now we consider the main case where $S \neq \emptyset$.

If $v \in T(\text{span}(S))$, then $v = T(w)$ for some $w \in \text{span}(S)$. By the definition of span, 

$$w = \sum_{i=1}^{n} a_i s_i$$

for some $n \geq 1$, some $a_i \in F$ and some $s_i \in S$. So, 

$$v = T(w) = T \left( \sum_{i=1}^{n} a_i s_i \right) = \sum_{i=1}^{n} a_i T(s_i) \in \text{span}(T(S))$$

Now suppose $v \in \text{span}(T(S))$. Then there exists $n \geq 1$, $a_i \in F$ and $u_i \in T(S)$ such that 

$$v = \sum_{i=1}^{n} a_i u_i.$$ 

Since each $u_i \in T(S)$, there exists $s_i \in S$ such that $u_i = T(s_i)$. So, 

$$v = \sum_{i=1}^{n} a_i u_i = \sum_{i=1}^{n} a_i T(s_i) = T \left( \sum_{i=1}^{n} a_i s_i \right) \in T(\text{span}(S))$$

So, $T(\text{span}(S)) = \text{span}(T(S))$. 

4. Consider $T: M_{n \times n}(F) \rightarrow M_{n \times n}(F)$, given by $T(A) = A - A^t$. Prove that $T$ is a linear transformation.

For all $A, B \in M_{n \times n}(F)$ and $c \in F$,

$$T(A + cB) = (A + cB) - (A + cB)^t$$

$$= A + cB - (A^t + (cB)^t)$$

$$= A + cB - (A^t + c(B^t))$$

$$= A - A^t + cB - c(B^t)$$

$$= T(A) + cT(B)$$

So, $T$ is a linear transformation.

5. Let $P_4(\mathbb{R})$ be the vector space of polynomials of degree less than or equal to 4 over the real numbers. Let

$$W = \{ f \in P_4(\mathbb{R}) \mid f(5) = 6 \cdot f(7) \}$$

(a) Prove that $W$ is a subspace of $V$.

Note, if you do part (b) first, then you can show that $W$ is the null space of $T$, hence a subspace. Here is how to do it directly.

For the zero polynomial, $f_0$ where $f_0(x) = 0$, $f_0(5) = 0 = 6 \cdot 0 = c \cdot f_0(7)$. So, $f_0 \in W$.

Let $f, g \in W$ and $c \in \mathbb{R}$. Then

$$(f + cg)(5) = f(5) + (cg)(5)$$

$$= f(5) + c \cdot g(5)$$

$$= 6f(7) + c \cdot 6g(7)$$

$$= 6(f(7) + c \cdot g(7))$$

$$= 6((f + cg)(7))$$

So, $f + cg \in W$. Thus, $W$ is a subspace.

(b) Let $T: P_4(\mathbb{R}) \rightarrow \mathbb{R}$ be given by $T(f) = f(5) - 6f(7)$. Prove $T \in (P_4(\mathbb{R}))^*$.

Since this is a function to $\mathbb{R}$ the field of scalars, we just have to show that $T$ is a linear transformation to have that it is a linear functional.

Now, suppose $f, g \in P_4(\mathbb{R})$ and $c \in \mathbb{R}$. Then

$$T(f + cg) = (f + cg)(5) - 6(f + cg)(7)$$

$$= f(5) + (cg)(5) - 6f(7) + (cg)(7))$$

$$= f(5) + c \cdot g(5) - 6(f(7) + c \cdot g(7))$$

$$= f(5) + c \cdot g(5) - 6f(7) - 6c \cdot g(7)$$

$$= f(5) - 6f(5) + c(g(5) - 6 \cdot g(7))$$

$$= T(f) + cT(g)$$

So, $T$ is a linear transformation, hence a linear functional, so $T \in (P_4(\mathbb{R}))^*$. 
(c) Determine, with proof, the nullity of $T$.

We will use rank-nullity. We know from class that $\dim(\mathbb{P}_4(\mathbb{R})) = 5$ ($1, x, \ldots, x^4$ is a basis). For the polynomial $x$, $T(x) = 5 - 6 \cdot 7 = -37 \neq 0$, so $\text{rank}(T) \geq 1$. But, the range is contained in $\mathbb{R}$ which is one dimensional. So, $\text{rank}(T) = 1$. By Rank-Nullity, we get that the nullity of $T$ is $\dim(\mathbb{P}_4(\mathbb{R})) - \text{rank}(T) = 5 - 1 = 4$.

6. Let $V$ be a vector space, and recall that if $S \subseteq V$, we define

$$S^0 = \{ f \in V^* \mid f(s) = 0 \text{ for all } s \in S \}.$$ 

(a) Prove that if $S_1 \subseteq S_2 \subseteq V$, then $S_2^0 \subseteq S_1^0$.

If $f \in S_2^0$, then for all $s \in S_1$, we have $s \in S_2$ since $S_1 \subseteq S_2$. But then $f(s) = 0$ since $f \in S_2^0$. So, $f \in S_1^0$. Therefore, $S_2^0 \subseteq S_1^0$.

(b) Prove that if $S \subseteq V$, then $S^0$ is a subspace of $V^*$.

Let $f, g \in S^0$, $c \in F$, and $s \in S$. Then

$$(f + cg)(s) = f(s) + cg(s) = 0 + c \cdot 0 = 0$$

So, $f + cg \in S^0$. Thus, $S^0$ is a subspace of $V^*$.

(c) Prove that if $S \subseteq V$, then $S^0 = \text{span}(S)^0$.

Since $S \subseteq \text{span}(S)$, we get $\text{span}(S)^0 \subseteq S^0$ by part (a).

Now if $f \in S^0$ and $x \in \text{span}(S)$, then there exists $n \geq 1$, $s_i \in S$ and $a_i \in F$ such that $x = \sum_{i=1}^{n} a_i s_i$. Then

$$f(x) = f \left( \sum_{i=1}^{n} a_i s_i \right) = \sum_{i=1}^{n} a_i f(s_i)$$

by linearity. But $f \in S^0$ and each $s_i \in S$, so this sum equals $\sum_{i=1}^{n} a_i \cdot 0 = 0$. Thus, $f \in \text{span}(S)^0$.

Therefore, $S^0 = \text{span}(S)^0$. 