

# Elliptic Curves over Real Quadratic Fields are Modular

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# Motivation

## Conjecture

*Let  $E$  be an elliptic curve over a totally real field  $K$ . Then  $E$  is modular in the following sense: there is a Hilbert eigenform  $f$  of parallel weight 2 over  $K$  such that  $L(E, s) = L(f, s)$ .*

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*All elliptic curves over  $\mathbb{Q}$  are modular.*

## Theorem (Jarvis and Manoharmayum 2004)

*Semistable elliptic curves over  $\mathbb{Q}(\sqrt{2})$  and  $\mathbb{Q}(\sqrt{17})$  are modular.*

## Prove Your Own Modularity Theorem

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$$\bar{\rho}_{E,p} : G_K \rightarrow \text{Aut}(E[p]) \cong \text{GL}_2(\mathbb{F}_p)$$

the representation giving the action of  $G_K$  on the  $p$ -torsion of  $E$ .

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## Definition

We say  $\overline{\rho}_{E,p}$  is **modular** if there exists a Hilbert cuspidal eigenform  $f$  over  $K$  of parallel weight 2, and a place  $\varpi \mid p$  of  $\overline{\mathbb{Q}}$  such that

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## Fact

$E$  modular  $\implies \overline{\rho}_{E,p}$  modular.



# A Theorem of Breuil and Diamond (2013)

**Théorème 3.2.2.** — Supposons  $p > 2$ ,  $\bar{\rho} : \text{Gal}(\bar{\mathbb{Q}}/F) \rightarrow \text{GL}_2(k_E)$  modulaire,  $\bar{\rho}|_{\text{Gal}(\bar{\mathbb{Q}}/F(\sqrt[5]{1}))}$  irréductible et, si  $p = 5$ , l'image de  $\bar{\rho}(\text{Gal}(\bar{\mathbb{Q}}/F(\sqrt[5]{1})))$  dans  $\text{PGL}_2(k_E)$  non isomorphe à  $\text{PSL}_2(\mathbb{F}_5)$ . Soit  $\psi : \text{Gal}(\bar{\mathbb{Q}}/F) \rightarrow E^\times$  un caractère qui relève  $\det \bar{\rho}$  et tel que  $\psi \varepsilon^{-1}$  est d'ordre fini,  $T$  un sous-ensemble de l'ensemble des places de  $F$  divisant  $p$  et  $S$  un ensemble fini de places finies de  $F$  contenant les places divisant  $p$  et les places où  $\bar{\rho}$  ou  $\psi$  sont ramifiés. Pour chaque  $v \in S$ , soit  $[r_v, N_v]$  un type de Weil-Deligne en  $v$  et pour chaque  $v \in T \cup \{v \nmid p, N_v \neq 0\}$ , soit  $\bar{\mu}_v : \text{Gal}(\bar{F}_v/F_v) \rightarrow k_E^\times$  un caractère. Supposons que, pour chaque  $v \in S$ ,  $\bar{\rho}|_{\text{Gal}(\bar{F}_v/F_v)}$  admet un relevé  $\rho_v : \text{Gal}(\bar{F}_v/F_v) \rightarrow \text{GL}_2(E)$  tel que :

- (i) si  $v|p$  alors  $\rho_v$  est potentiellement semi-stable de poids de Hodge-Tate  $(0, 1)$  pour tout  $F_v \hookrightarrow \bar{\mathbb{Q}}_p$
- (ii) si  $v|p$  alors  $\rho_v$  est potentiellement ordinaire si et seulement si  $v \in T$
- (iii)  $\rho_v$  est de type de Weil-Deligne  $[r_v, N_v]$  ( $v \in S$ )
- (iv) si  $v \in T \cup \{v \nmid p, N_v \neq 0\}$  alors  $\rho_v$  a une sous-représentation  $\sigma_v$  de dimension 1 telle que  $\sigma_v$  relève  $\bar{\mu}_v \omega$  et  $\sigma_v \varepsilon^{-1}|_{I_v}$  est d'ordre fini
- (v)  $\det \rho_v|_{I_v} = \psi|_{I_v}$  ( $v \in S$ ).

Alors, quitte à agrandir  $E$ ,  $\bar{\rho}$  possède un relevé  $\rho : \text{Gal}(\bar{\mathbb{Q}}/F) \rightarrow \text{GL}_2(E)$  continu non ramifié en dehors de  $S$  et tel que :

- (i) si  $v|p$  alors  $\rho|_{\text{Gal}(\bar{F}_v/F_v)}$  est potentiellement semi-stable de poids de Hodge-Tate  $(0, 1)$  pour tout  $F_v \hookrightarrow \bar{\mathbb{Q}}_p$
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De plus, un tel relevé  $\rho$  de  $\bar{\rho}$  provient d'une forme modulaire de Hilbert de poids  $(2, 2, \dots, 2)$ .

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*Let  $p \geq 3$ . Write  $\bar{\rho} = \bar{\rho}_{E,p}$ . Suppose*

- (i)  $\bar{\rho}$  is modular,*
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## Corollary

*If  $E$  satisfies the Big Image Condition mod 3 then  $E$  is modular.*

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## Fact

*If  $E$  violates the Big Image Condition mod  $p$ , then  $E$  gives rise to a  $K$ -point on  $X_0(p)$ ,  $X_{\text{ns}}(p)$  or  $X_s(p)$ .*

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## Example

The maps  $X_0(3) \rightarrow X(1)$ ,  $X_{\text{ns}}(3) \rightarrow X(1)$  and  $X_s(3) \rightarrow X(1)$  are given by

$$t \mapsto \frac{(t+27)(t+243)^3}{t^3}, \quad t \mapsto t^3, \quad t \mapsto \frac{(t-9)^3(t+3)^3}{t^3}.$$



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## Corollary

Let  $j$  be the  $j$ -invariant of  $E$ . If

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## Conclusion

There are infinitely many  $j$ -invariants  $\in K$  for which we cannot yet lift modularity of  $\bar{\rho}_{E,3}$ .

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Let  $p \neq 2, 3$ . We **TRY** to show

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To make this work, need 'lots' of  $K$ -points on  $X_E(p)$ .

$$\text{genus}(X_E(p)) = \begin{cases} 0 & p = 5 \\ \geq 3 & p \geq 7 \end{cases}.$$

**Conclusion:** Modularity switching as above works for  $p = 5$  but not 7.

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**Problem:**  $X_0(3) \times_{X(1)} X_0(5) \cong X_0(15)$  has genus 1, and  $X_0(15)(K)$  could be infinite. So there might still be infinitely many non-modular  $j \in K$ .

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### Theorem (Langlands Solvable Base Change)

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### Theorem (Manoharmayum, Freitas–Le Hung–S.)

If  $E/K$  satisfies the Big Image Condition mod 7 then  $E$  is modular.

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## Theorem (Calegari, Freitas–Le Hung–S.)

*There are at most finitely many  $j$ -invariants of elliptic curves over  $K$  that are non-modular.*

## Modularity Continued

To prove modularity for all real quadratic fields, it is enough to compute all the non-cuspidal real quadratic points on

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A much finer analysis shows that it enough to do this for the following seven modular curves:

- $X(b_5, b_7)$  (genus 3);
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- $X(s_3, s_5)$  (genus 4);
- $X(b_3, b_5, d_7)$  (genus 97);
- $X(s_3, b_5, d_7)$  (genus 153);
- $X(b_3, b_5, e_7)$  (genus 73);
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$b = \text{borel}$ .

$s = \text{normalizer of split Cartan}$ .

$d_7$  has image  $\cong D_3$  in  $\text{PGL}_2(\mathbb{F}_7)$ .

$e_7$  has image  $\cong D_4$  in  $\text{PGL}_2(\mathbb{F}_7)$ .

$$X(b5, b7) = X_0(35)$$

$$X_0(35) : y^2 = (x^2 + x - 1)(x^6 - 5x^5 - 9x^3 - 5x - 1).$$

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If  $P$  is a quadratic point on  $X_0(35)$ , then

$$[P + P^\sigma - \infty_+ - \infty_-] \in J_0(35)(\mathbb{Q}).$$

### Lemma

All quadratic points on  $X_0(35)$  have the form

$$P = (x, \pm\sqrt{f(x)}), \quad f(x) = (x^2 + x - 1)(x^6 - 5x^5 - 9x^3 - 5x - 1)$$

with  $x \in \mathbb{Q}$  (except for  $(\frac{-1 \pm \sqrt{5}}{2}, 0)$ ).



## Modular Interpretation of Real Quadratic $P$

$$P = \left( x, \sqrt{f(x)} \right) = (E, C), \quad x \in \mathbb{Q}, \quad K = \mathbb{Q}(\sqrt{f(x)})$$

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**Moral:** If you want to prove modularity of quadratic points on a modular curve  $X$ , use Mordell–Weil information (over  $\mathbb{Q}$ ) to prove that Galois conjugation is a geometric involution on  $X$ .



## A Big Example

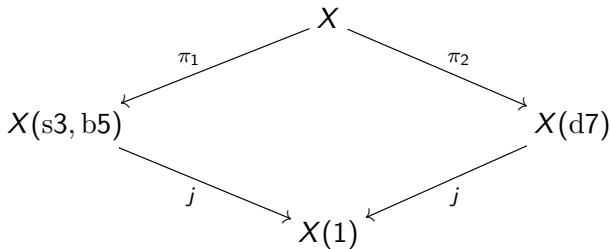
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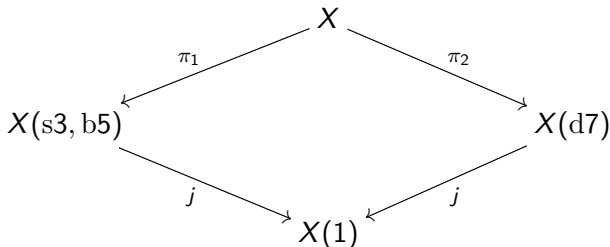
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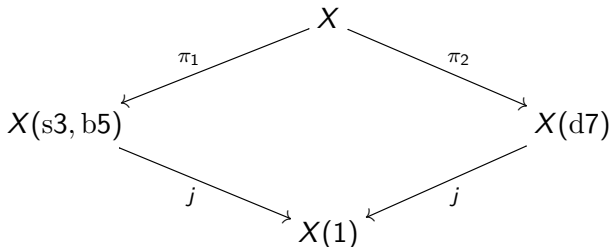
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**Representing points on  $X$ :** Roughly speaking, if  $\mathbb{F}$  is a field, then  $P \in X(\mathbb{F})$  is a pair  $(P_1, P_2)$  where  $P_1 \in X(s3, b5)(\mathbb{F})$  and  $P_2 \in X(d7)(\mathbb{F})$  with  $j(P_1) = j(P_2)$ .

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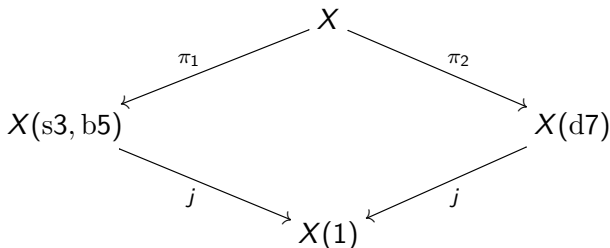
### Mordell–Weil Information

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### Mordell–Weil Information

$$X(s3, b5) = 15A3,$$

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Moreover,  $X(s3, b5)(\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ ,  $X(d7)(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ .

$$P \in X(K)$$

$$P \in X(K) \implies Q := \pi_2(P) \in X(d7)(K)$$



$$\begin{aligned} P \in X(K) &\implies Q := \pi_2(P) \in X(d7)(K) \\ &\implies Q + Q^\sigma \in X(d7)(\mathbb{Q}) = \{\mathcal{O}, T\}. \end{aligned}$$

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**Objective:** Show that this is true for all  $P \in X(K)$  for all quadratic  $K$ .

## The Mordell–Weil Sieve

$$\begin{array}{ccc} X^{(2)}(\mathbb{Q}) & \xrightarrow{\alpha} & X(s7, b5)(\mathbb{Q}) \times X(d7)(\mathbb{Q}) \\ \downarrow & & \downarrow \mu \\ \prod X^{(2)}(\mathbb{F}_p) & \xrightarrow{\beta} & \prod X(s7, b5)(\mathbb{F}_p) \times X(d7)(\mathbb{F}_p) \end{array}$$

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Using  $11 \leq p \leq 100$  we find

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Note  $\pi_2(P) + \pi_2(P)^\sigma = 0$ .

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Note  $\pi_2(P) + \pi_2(P)^\sigma = 0$ . So  $P$  is modular!!

Theorem (Freitas–Le Hung–S.)

*Let  $E$  be an elliptic curve over a real quadratic field  $K$ . Then  $E$  is modular.*

Thank You!