L-functions with rational integer coefficients

Sally Koutsoliotas
Bucknell University

joint work with
David W. Farmer, *American Institute of Mathematics*
Stefan Lemurell, *Gothenburg University &*
Chalmers University of Technology

March 14, 2014
Curves and Automorphic Forms
Arizona State University

Wanted: table of hyperelliptic curves

a. ordered by conductor, N

b. certain none is missing

Available data:
M. Stoll

169	$y^2 = x^6 + 4x^5 + 6x^4 + 2x^3 + x^2 + 2x + 1$
249	$y^2 = x^6 + 2x^3 + 4x^2 + 4x + 1$
277	$y^2 = 4x^5 - 8x^3 + 8x^2 - 4x + 1$
295	$y^2 = x^6 - 4x^4 - 2x^3 + 1$

Stoll's list does not attempt to be complete.

From Brumer and Kramer:

Table 2. Paramodular Abelian Surfaces of ODD Conductor < 1000.

N	EQUATION	INFO
249	$x^6 + 4x^5 + 4x^4 + 2x^3 + 1$	83
277	$x^6 + 2x^5 + 3x^4 + 4x^3 - x^2 - 2x + 1$	q
295	$x^6 - 2x^3 - 4x^2 + 1$	59
349	$x^6 - 2x^5 + 3x^4 - x^2 - 2x + 1$	q
353	$x^6 + 2x^5 + 5x^4 + 2x^3 + 2x^2 + 1$	wr72
389	$x^6 + 2x^5 + 5x^4 + 8x^3 + 8x^2 + 4x$	389
427	$x^6 - 4x^5 - 4x^4 + 18x^3 + 16x^2 - 16x - 15$	61
461	$x^6 + 2x^5 - 5x^4 - 8x^3 + 11x^2 + 10x - 11$	q
523	$x^6 - 2x^5 + x^4 + 4x^3 - 4x^2 - 4x$	523
555	$x^6 + 6x^5 + 5x^4 - 16x^3 - 8x^2 + 12x$	37
561	PRYM	11, 51
587a	$-3x^6 + 18x^4 + 6x^3 + 9x^2 - 54x + 57$	S_6, mild@3
587b	$x^6 + 2x^4 + 2x^3 - 3x^2 - 2x + 1$	S_6
597	$x^6 + 4x^5 + 8x^4 + 12x^3 + 8x^2 + 4x$	q
603	$x^6 - 4x^5 + 2x^4 + 4x^3 + x^2 - 4x$	67
623	$-224x^6 - 1504x^5 - 4448x^4 - 7200x^3 - 6080x^2 - 2048x$	89, mild@8
633	$24x^6 + 40x^5 + 28x^4 + 80x^3 + 52x^2 - 32x$	211, mild@2
657	WEIL RESTRICTION	u, notSS
665	PRYM	19, 35
691	$x^6 + 2x^5 - 3x^4 - 4x^3 + 4x$	691
709	$-4x^5 - 7x^4 - 4x$	709
713a	$x^6 + 2x^5 + x^4 + 2x^3 - 2x^2 + 1$	23, 31
713b	$x^6 - 2x^5 + x^4 + 2x^3 + 2x^2 - 4x + 1$	23, 31

What is already known:

Mestre *everything**	Stoll *hyperelliptic curves*	Brumer & Kramer *abelian surfaces*
none with $N < 107$		
	169	
	249	249
	277	277
	295	295

Goal: make a complete list up to N_{max}, say 300.

1. prove there are no other conductors, $N \leq N_{max}$
2. for each conductor found, find all the isomorphism classes

L-function axioms

- Dirichlet series $L(s) = \sum_{n=1}^{\infty} \frac{A_n/\sqrt{n}}{n^s}$ $A_n \in \mathbb{Z}$
- Functional equation

$$\Lambda(s) := N^{s/2}\, \Gamma_{\mathbb{C}}(s + \tfrac{1}{2})^2\, L(s) \;=\; \pm\, \Lambda(1-s)$$

- Euler product $L(s) = \prod_p F_p(p^{-s})^{-1}$, where

$$F_p(z) = G_p(z/\sqrt{p}) \text{ with } G_p(z) \in \mathbb{Z}[z]$$

L-function axioms

- Dirichlet series $\quad L(s) = \sum_{n=1}^{\infty} \frac{A_n/\sqrt{n}}{n^s} \qquad A_n \in \mathbb{Z}$
- Functional equation

$$\Lambda(s) := N^{s/2}\,\Gamma_{\mathbb{C}}(s+\tfrac{1}{2})^2\,L(s) \;=\; \pm\,\Lambda(1-s)$$

- Euler product $L(s) = \prod_p F_p(p^{-s})^{-1}$, where

$$F_p(z) = G_p(z/\sqrt{p}) \text{ with } G_p(z) \in \mathbb{Z}[z]$$

  Furthermore
  - $F_p(0) = 1$
  - If $p \nmid N$ then $F_p(z)$ has degree 4.
    If $p|N$ then $F_p(z)$ has degree ≤ 3.
  - If $p \nmid N$ then all roots of $F_p(z)$ lie on $|z| = 1$.
  - If $p|N$ then each root of $F_p(z)$ lies on $|z| = p^{m/2}$ for some $m \in \{0, 1, 2, 3\}$.

Conclusion: there are only finitely many choices for F_p for each p.

Number of possible local factors

prime, p	good	bad
2	35	26
3	63	32
5	129	38
7	207	44

The Method:

For a given N and $\varepsilon = \pm 1$, everything about the L-function is known *except the coefficients.*

$\Longrightarrow$ find coefficients such that the functional equation is true

OR

prove that the L-function cannot exist.

The Approximate Functional Equation

$g(s)$: a nice test function

$$g(s)\,\Lambda(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \int_{(\nu)} f_{compl.}(g) + \sum_{n=1}^{\infty} \frac{\overline{a_n}}{n^{1-s}} \int_{(\nu)} f_{compl.}(g)$$

Solving for $L(s)$ gives

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \int_{(\nu)} h_{compl.}(g) + \sum_{n=1}^{\infty} \frac{\overline{a_n}}{n^{1-s}} \int_{(\nu)} h_{compl.}(g)$$

Choose s_0 and $g_1(s)$:

$$L(s_0) = \bigstar + \bigstar\ a_2 + \bigstar\ a_3 + \bigstar\ a_4 + \bigstar\ a_5 + \bigstar\ a_6 + \ldots$$

Example: $g(s) = e^{\frac{7}{8}s}$, if $f \in S_{24}(\Gamma_0(1))$ then

$$L(\tfrac{1}{2}, f) = 1.473a_1 + 1.186a_2 - 0.0959a_3 - 0.00772a_4 + 0.000237a_5 + \cdots.$$

Our Method

$$s_0, g_1 : L(s_0) = \star + \star\ a_2 + \star\ a_3 + \star\ a_4 + \star\ a_5 + \star\ a_6 + \dots$$

$$s_0, g_2 : L(s_0) = \star + \star\ a_2 + \star\ a_3 + \star\ a_4 + \star\ a_5 + \star\ a_6 + \dots$$

Subtracting gives:

$$0 = \star + \star\ a_2 + \star\ a_3 + \star\ a_4 + \star\ a_5 + \star\ a_6 + \dots$$

Apply Ramanujan bounds for a_j and check whether LHS $=$ RHS.

Example: no hyperelliptic curve with $N = 125$ and $\varepsilon = +1$

Choose $s_0 = \frac{1}{2}$. Using the test function $g(s) = 1$:

$$L(\tfrac{1}{2}) = 0.123 + 0.0241a_2 + 0.0058a_3 + 0.00195a_4 + 0.00075a_5 + \cdots$$

and with $g(s) = e^{\frac{1}{4}s}$:

$$L(\tfrac{1}{2}) = 0.116 + 0.01834a_2 + 0.00444a_3 + 0.0013a_4 + 0.00044a_5 + \cdots .$$

Subtracting and rescaling:

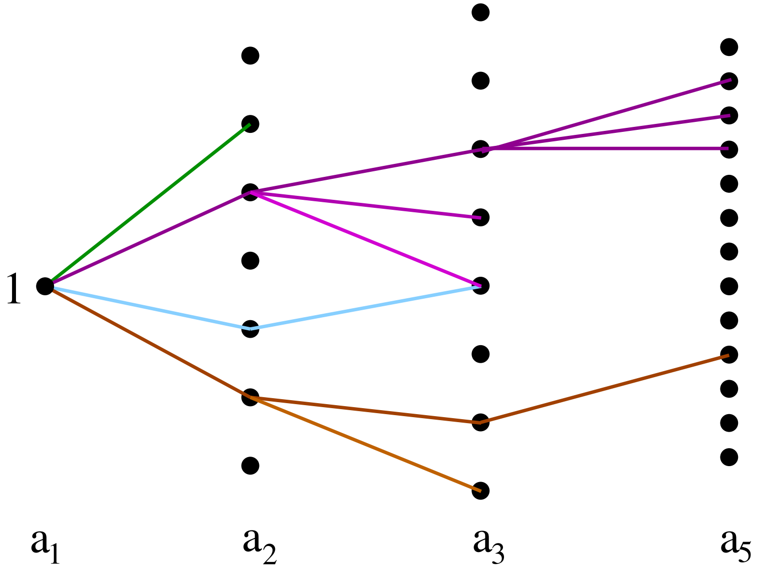
$$0 = 1 + 0.427a_2 + 0.187a_3 + 0.087a_4 + 0.043a_5 + 0.022a_6 + \cdots .$$

Try $F_2(x) = 1 - 2\sqrt{2}x + 4x^2 - 2\sqrt{2}x^3 + x^4$:

$$0 = 2.57 + 0.254a_3 + 0.049a_5 + 0.0130a_7 + 0.00399a_9 + \cdots .$$

The Ramanujan bound is $|a_p| \leq 4$.
$\Longrightarrow$ Contradiction. (34 more choices to try for F_2.)

Searching a tree:



Results:

For $N \leq 300, \varepsilon = +1$

eliminate all except:

121	154	165	169	187	196	209	210	220
225	231	238	249	255	256	264	266	277
280	285	286	289	294	295	297	300	

Results:
For $N \le 300, \varepsilon = +1$

eliminate all except:

121	154	165	169	187	196	209	210	220
225	231	238	249	255	256	264	266	277
280	285	286	289	294	295	297	300	

$$N = 121: \qquad L(s, E_{11}) \quad L(s, E_{11})$$

Results:
For $N \leq 300, \varepsilon = +1$

eliminate all except:

121	154	165	169	187	196	209	210	220
225	231	238	249	255	256	264	266	277
280	285	286	289	294	295	297	300	

$N = 121:$ $L(s, E_{11})$ $L(s, E_{11})$
$N = 154:$ $L(s, E_{11})$ $L(s, E_{14})$
$N = 286:$ $L(s, E_{11})$ $L(s, E_{26})$ (two distinct L-functions)

Results:

For $N \leq 300, \varepsilon = +1$

eliminate all except:

121	154	165	169	187	196	209	210	220
225	231	238	249	255	256	264	266	277
280	285	286	289	294	295	297	300	

product of two elliptic curve L-functions over $\mathbb{Q}$

Results:
For $N \leq 300, \varepsilon = +1$

eliminate all except:

121	154	165	169	187	196	209	210	220
225	231	238	249	255	256	264	266	277
280	285	286	289	294	295	297	300	

product of two elliptic curve L-functions over $\mathbb{Q}$

hyperelliptic curve/product of conjugate cuspforms

$N = 169$

hyperelliptic curve: $y^2 = x^6 + 4x^5 + 6x^4 + 2x^3 + x^2 + 2x + 1$
(Stoll)

But L-function is not primitive–it arises from two conjugate cuspforms of level 13:

$$L(s) = L(s, f)\ L(s, \overline{f}).$$

Specifically,

$$f_1, f_2 \in S_2(\Gamma_0(13), \chi_{13}(4, \cdot)) \quad \text{2-dimensional}$$

where χ is the Dirichlet character defined by $\chi_{13}(4, 2) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Now,

$$f_1(q) = q + \left(\frac{-3 + i\sqrt{3}}{2}\right) q^2 + \left(-1 + i\sqrt{3}\right) q^3 + \cdots$$

$$f_2(q) = q + \left(\frac{-3 - i\sqrt{3}}{2}\right) q^2 + \left(-1 - i\sqrt{3}\right) q^3 + \cdots$$

We find, in arithmetic normalisation,

$$L(s) = 1 - \frac{3}{2^s} - \frac{2}{3^s} + \cdots$$

which agrees with the data from the LMFDB up to a_{19}.

Results:
For $N \le 300, \varepsilon = +1$

eliminate all except:

121	154	165	169	187	196	209	210	220
225	231	238	249	255	256	264	266	277
280	285	286	289	294	295	297	300	

product of two elliptic curve L-functions over $\mathbb{Q}$
hyperelliptic curve/product of conjugate cuspforms
hyperelliptic curve/abelian surface/paramodular form
(Stoll; Brumer and Kramer; Poor and Yuen)

Results:

For $N \le 300, \varepsilon = +1$

eliminate all except:

121	154	165	169	187	196	209	210	220
225	231	238	249	255	256	264	266	277
280	285	286	289	294	295	297	300	

product of two elliptic curve L-functions over $\mathbb{Q}$
hyperelliptic curve/product of conjugate cuspforms
hyperelliptic curve/abelian surface/paramodular form
(Stoll; Brumer and Kramer; Poor and Yuen)

What about $N = 256$?

$N = 256$

The L-function is not primitive; arises from two conjugate cuspforms of level 16.

$$L(s) = L(s, f)\ L(s, \overline{f})$$

Now,

$$f_1, f_2 \in S_2(\Gamma_0(16), \chi_{16}(11, \cdot)) \quad \text{2-dimensional}$$

where χ is the Dirichlet character defined by

$$\chi_{16}(11, 5) = i,\ \ \chi_{16}(11, 15) = 1$$

and q-expansion:

$$q + (-1 \mp i)q^2 + (-1 \pm i)q^3 \pm 2iq^4 + (-1 \mp i)q^5 + 2q^6 + \cdots .$$

Does it come from a hyperelliptic curve?

Computational Theorem: For $N \leq 300$ and $\varepsilon = +1$, the only levels for which there exist L-functions are those that were previously known.

Computational Theorem: For $N \leq 450$ and $\varepsilon = -1$, the only level for which there exists an L-function is 407.

$N = 407$ arises from $L(s, E_{11})\ L(s, E_{37})$.

Final Questions

1. Does $N = 256$ come from a variety?

2. Why are there more non-primitive L-functions?

3. Why are small levels more likely to be odd?

4. Do all possible local factors eventually arise?

Stoll's search method

$$y^2 = x^6 + c_5x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + 1$$

1. loop over curves where c_j are small ($|c_j| \leq 5$)
2. find the discriminant Δ; continue if not too big
3. determine the conductor, N
4. save the curve if N is small

Only N odd cases.

Curves with small N but large coefficients will be absent from list.