Modular Forms and Elliptic Curves Over the Cubic Field of Discriminant -23

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Curves and Automorphic Forms (ASU)

Overview

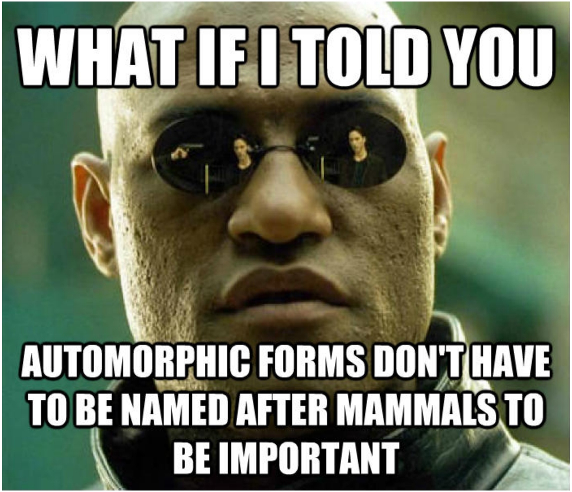
Let F be the cubic field of discriminant -23 and let $\mathcal{O} \subset F$ be its ring of integers. By explicitly computing cohomology of congruence subgroups of $\mathrm{GL}_2(\mathcal{O})$, we computationally investigate modularity of elliptic curves over F.

- (joint with Gunnells) *Modular forms and elliptic curves over the cubic field of discriminant* -23, Int. J. Number Theory 9 (2013), no. 1, 53-76.
- (joint with Donnelly, Gunnells, Klages-Mundt) in progress.

"Modular forms" . . .

- ". . . labels for different types of modular forms: Holomorphic, Hilbert, Bianchi, Siegel."
- ". . . talks that describe work on computations of elliptic curves over number fields and . . . the conjecturally corresponding modular forms (e.g., Hilbert, Siegel, Bianchi)"
- "misc."

“Modular forms” . . .



Motivation

$F = \mathbb{Q}$:

- Cusp forms can be described cohomologically

$$H^1(\Gamma_0(N)\backslash\mathfrak{h};\mathbb{C}) \simeq S_2(N) \oplus \overline{S}_2(N) \oplus \mathrm{Eis}_2(N).$$

- There is a link between elliptic curves and cusp forms

$$a_p(f) = p + 1 - \#E(\mathbb{F}_p).$$

Tessellation of $\mathfrak{h}$

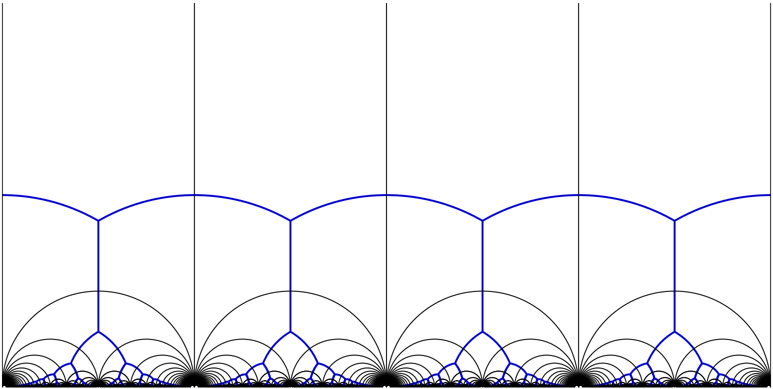


Figure : Well-rounded binary quadratic forms (blue) with dual tessellation by ideal triangles (black). Compute cohomology using topological techniques.

Modular symbols

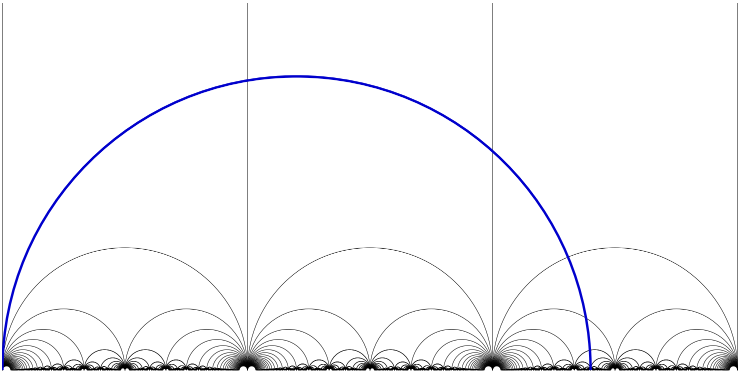


Figure : Compute Hecke operators: Unimodular symbols (black) and non-reduced symbol (blue).

Reduction of modular symbol

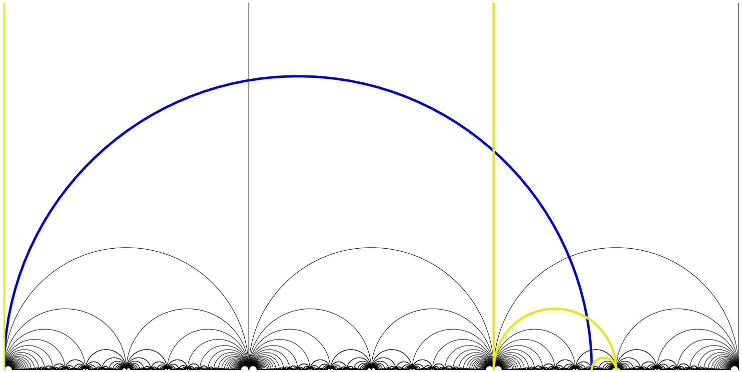


Figure : Re-expression of non-reduced symbol (blue) as a sum of unimodular symbols (gold) using geometric techniques.

Setting

Let F be a number field of class number 1 with ring of integers $\mathcal{O}$.

$$\begin{aligned}
\mathbf{G} &= \mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_n \\
G &= G(\mathbb{R}) \simeq \left(\prod \mathrm{GL}_n(\mathbb{R})\right) \times \left(\prod \mathrm{GL}_n(\mathbb{C})\right) \\
X &= \text{associated symmetric space } G/KA_{\mathbf{G}} \\
\Gamma &\subseteq \mathrm{GL}_n(\mathcal{O}), \text{ congruence subgroup}
\end{aligned}$$

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Which n and F?

$F = \mathbb{Q}$

- $n = 2$: Classical modular forms
- $n = 3$: Ash-Doud-Pollack, Ash-Grayson-Green, Ash-McConnell, van Geemen-van der Kallen-Top-Verberkmoes, ...
- $n = 4$: Ash-Gunnells-McConnell
- $n = 5, 6, 7$: Elbaz-Vincent-Gangl-Soulé (level 1, no Hecke)

Computations introducing level structure (no Hecke) for $n = 3, 4, 5, 6$ in progress.
(Ash-Elbaz-Vincent-Gunnells-McConnell-Pollack-Y)

Which n and F? (ctd.)

F a complex quadratic field

- $n = 2$: Grunewald, Cremona, Schwermer, Vogtmann, Sengun, Y, ...
- $n = 3, 4$: Dutour-Gangl-Gunnells-Hanke-Schürmann-Y (level 1, no Hecke)

F a totally real field

- $n = 2$: Dembélé, Voight, ...

F a CM quartic field ($\mathbb{Q}(\zeta_5)$)

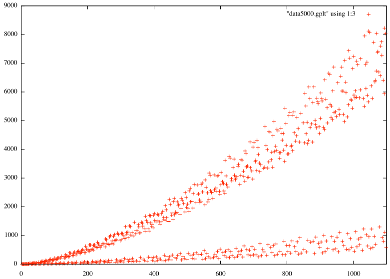
- $n = 2$: Gunnells-Hajir-Y

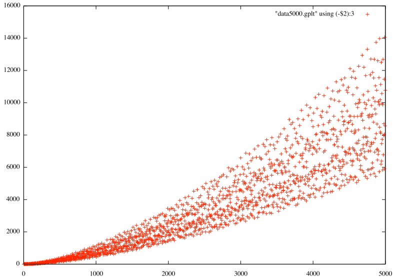
Why this particular F?

Let F be the cubic field of discriminant -23. Then

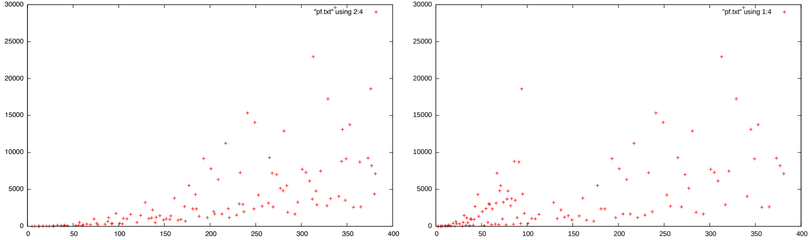
- F, n is the the right size: $n \cdot \deg(F)$ not large.
- F is a new type: not totally real or CM.
- $\mathrm{disc}(F)$ is not large. (Not necessary, but helps. See next slide.)

Number of perfect forms





Number of perfect forms



Cohomology

Borel conjectured and Franke proved that the complex cohomology of any arithmetic group can be computed in terms of certain automorphic forms.

Cohomology

For Γ torsion-free, the quotient $\Gamma\backslash X$ is a Eilenberg-Mac Lane space.

$$H^*(\Gamma;\mathbb{C}) \simeq H^*(\Gamma\backslash X;\mathbb{C}).$$

These are the cohomology spaces that are built from certain automorphic forms.

- Replace $\mathbb{C}$ with complex representation of $\mathbf{G}(\mathbb{Q})$ to introduce weight structure.
- Isomorphism is true even if Γ has torsion.

General setting

Voronoi, generalized by Ash and Koecher:

$$
\begin{aligned}
G &\simeq \prod \mathrm{GL}_n(\mathbb{R}) \times \prod \mathrm{GL}_n(\mathbb{C}) \\
V &= \prod V_v, \text{ where} \\
&\qquad V_v = \begin{cases} \mathrm{Sym}_n(\mathbb{R}) & \text{if } v \text{ is real,} \\ \mathrm{Herm}_n(\mathbb{C}) & \text{if } v \text{ is complex.} \end{cases} \\
C &= \prod C_v, \text{ where } C_v = V_v^+ \\
X &= C/\sim
\end{aligned}
$$

Natural G-action on C descends to G-action on X.

Space of forms

- View point A of C as form

$$A(x) = \sum_v x_v^* A_v x_v.$$

- For $A \in C$,

$$m(A) = \min\{A(x) \mid x \in \mathcal{O}^n \setminus 0\}$$
$$M(A) = \{x \in \mathcal{O}^n \mid A(x) = m(A)\}.$$

A form A is *perfect* if A is uniquely determined by $m(A)$ and $M(A)$.

"Cusps"

View point $x \in \mathcal{O}^n$ as point $q(x) \in \bar{C}$

$$q(x)_v = x_v x_v^*.$$

- $q(\mathcal{O}^n)$ is discrete in $\bar{C}$.

Koecher flag

Theorem

There is a Koecher flag Σ of polyhedral cones so that

$$\bigcup_{\sigma \in \Sigma} \sigma \cap C = C.$$

The Koecher flag gives a reduction theory in the following sense.

1. *There are finitely many Γ-orbits in Σ.*
2. *Each $y \in C$ is contained in a unique cone in Σ.*
3. *For each $\sigma \in \Sigma$ with $\sigma \cap C \neq \emptyset$, the stabilizer of σ is finite.*

The codimension 0 cones in Σ can be described in terms of perfect forms over F.

Setting

- $n = 2$
- $F =$ cubic field of discriminant -23
- $\mathbf{G}(\mathbb{R}) \simeq \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{C})$
- $X = \mathfrak{h} \times \mathfrak{h}_3 \times \mathbb{R}$

Discriminant -23

F number field of discriminant -23 defined by polynomial $x^3 - x^2 + 1$.

There are nine $\mathrm{GL}_2(\mathcal{O})$-classes of 6-polytopes.

- Seven are simplicial with f-vector $(7, 21, 35, 35, 21, 7)$.
- One has f-vector $(8, 28, 56, 68, 48, 16)$.
- One has f-vector $(9, 36, 81, 108, 81, 27)$.

Voronoi cells for F

Dimension	# of classes
6	9
5	35
4	47
3	31
2	10
1	1(1)
0	(1)

The number of $GL_2(\mathcal{O})$-classes of Voronoi cells are given.

Primes dividing order of stabilizers are $\{2, 3\}$.

Sharbly complex

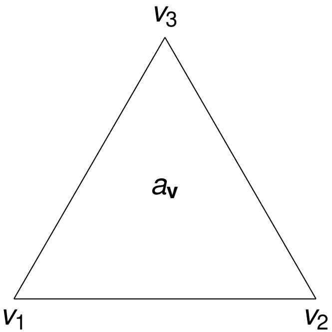
The *sharbly complex* $S_*(\Gamma)$ (Ash, Gunnells, Lee-Szczarba)

$$H^{\nu-k}(\Gamma;\mathbb{C}) \simeq H_k(S_*(\Gamma))$$

can be used describe Hecke action.

The cuspidal forms can be described by classes in $H_1(S_*(\Gamma))$.

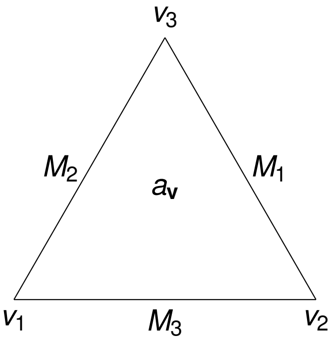
1-sharblies



For $n = 2$, 1-sharblies are formal sums of triples of vertices of Π

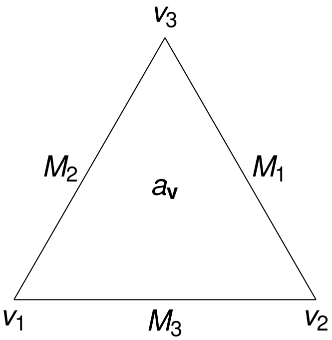
$$\sum a_{\mathbf{v}}[v_1, v_2, v_3].$$

1-sharbly cycle

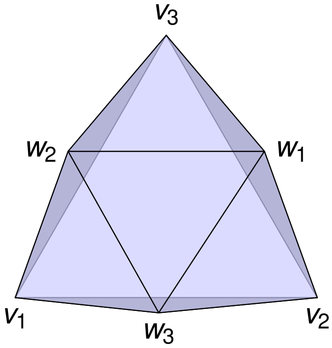


- $a_{\mathbf{v}}$ = coefficient
- v_i = vertices
- M_i = lift data
- boundary vanishes modulo Γ

Generic case (3 bad edges)

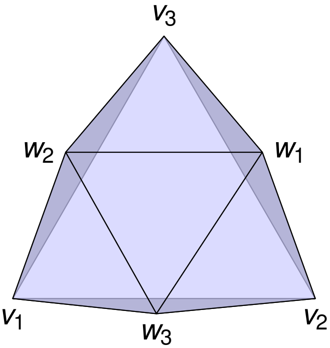


$\longrightarrow$



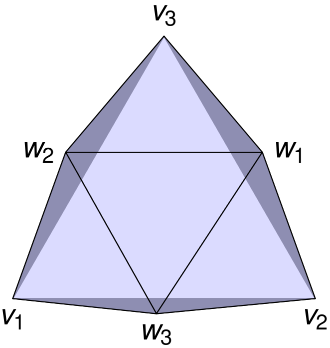
Nature of the new 1-sharblies

How do the new 1-sharblies compare to the original?



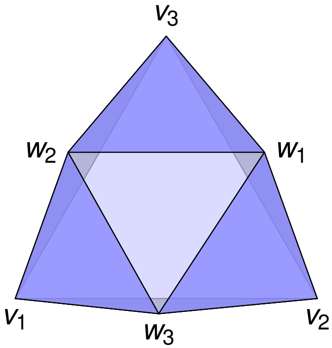
Nature of the new 1-sharblies

Vanish modulo Γ since reducing is Γ-equivariant.



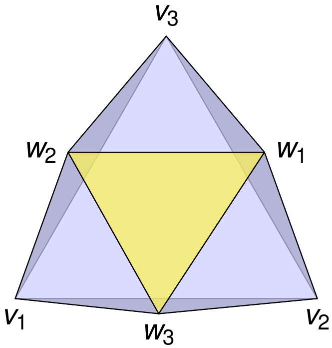
Nature of the new 1-sharblies

"Better" than original by choice of reduction point.

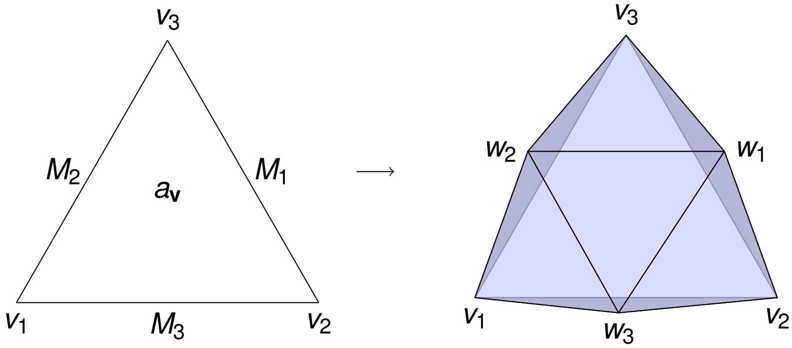


Nature of the new 1-sharblies

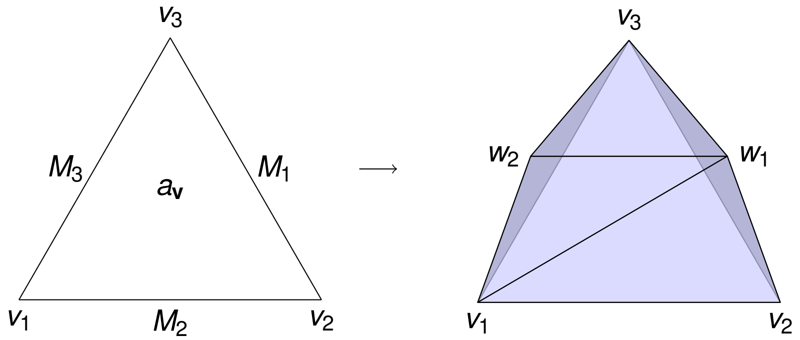
"Should be" better than original.



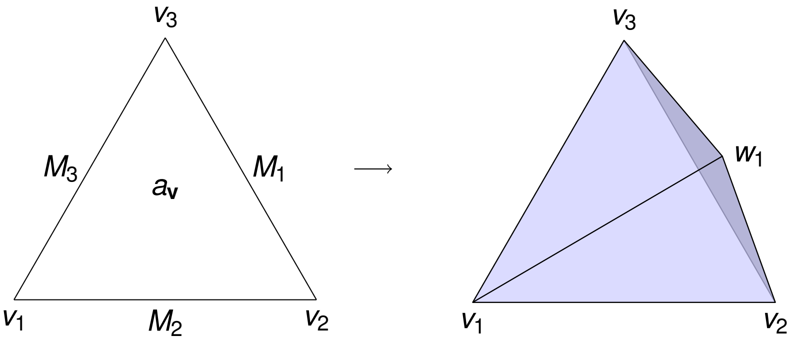
Generic case (3 bad edges)



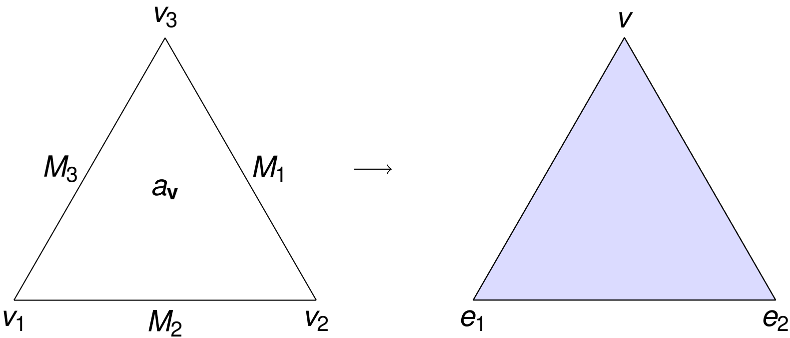
Less generic case (2 bad)



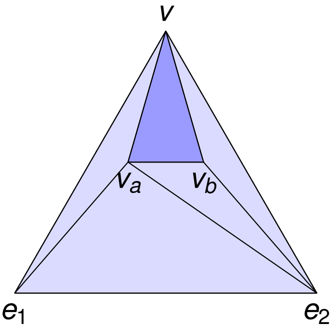
Even less generic case (1 bad)

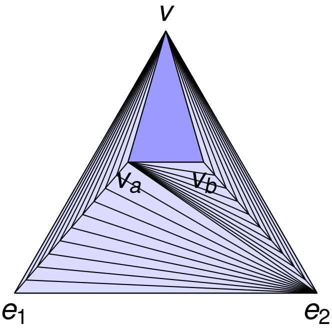


Very special case (0 bad)



Very special case (0 bad)





Data

Example: For level $(4t^2 - t - 5)$ of norm 89, the cuspidal space is 1-dimensional. We find an elliptic curve

$$[a_1, a_2, a_3, a_4, a_6] = [t-1, -t^2-1, t^2-t, t^2, 0]$$

with

$$|E(\mathbb{F}_{\mathfrak{p}})| = \mathrm{N}(\mathfrak{p}) + 1 - a_{\mathfrak{p}}.$$

Summary of first computation

- 44 levels with 1-dimensional rational Hecke eigenspace: We found a matching elliptic curve over F.
- For no level/conductor of norm ≤ 835 did we find discrepancies.
- 10 levels with 2-dimensional eigenspace on which the Hecke operators acted by rational scalar matrices. These were “old” cohomology classes and can be accounted for by cohomology classes at lower levels.
- At norm level $529 = 23^2$, we found a two-dimensional cuspidal subspace where the eigenvalues live in $\mathbb{Q}(\sqrt{5})$. The eigenvalues of this cohomology class match those of the weight two newform of level 23.

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More examples . . .

Nm	a_1	a_2	a_3	a_4	a_6
89	$t-1$	$-t^2-1$	t^2-t	t^2	0
107	0	$-t$	$-t-1$	$-t^2-t$	0
115	$-t^2+t-1$	$-t^2+1$	$t-1$	-1	$-t^2$
136	$-t^2$	-1	$-t^2+1$	$t+1$	0
161	t^2-t-1	$-t^2+t-1$	t^2-t+1	t^2-t	$t-1$
167	t^2+1	$t+1$	t^2+t-1	$-t^2-t+1$	$-t^2+t+1$
185	t	$-t^2+t+1$	$t+1$	0	0
223	1	t^2	t^2+t-1	$-t^2+t-1$	1
253	-1	$-t^2-t$	$-t^2-t$	$-t^2-t$	0
259	0	1	$-t^2-t-1$	t^2-t+1	$-t^2-t+1$
275	$-t^2+t$	t	t^2-t	0	0
289	-1	t^2-t	t	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Data for isogeny classes of elliptic curves up to norm conductor 11575

- 3067 isomorphism classes in 1250 isogeny classes
- isogeny classes, by rank: [470, 772, 8]
- isomorphism classes, by rank: [1644, 1414, 9]
- isogeny classes where rank is not proved: [0, 241, 1]
- 250 levels with leftover newspace

Torsion subgroups of E

structure	# isom	structure	# isom
0	688	$\mathbb{Z}/8\mathbb{Z}$	29
$\mathbb{Z}/2\mathbb{Z}$	1163	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	77
$\mathbb{Z}/3\mathbb{Z}$	209	$\mathbb{Z}/9\mathbb{Z}$	6
$\mathbb{Z}/4\mathbb{Z}$	293	$\mathbb{Z}/10\mathbb{Z}$	20
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	245	$\mathbb{Z}/12\mathbb{Z}$	8
$\mathbb{Z}/5\mathbb{Z}$	53	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	16
$\mathbb{Z}/6\mathbb{Z}$	237	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$	5
$\mathbb{Z}/7\mathbb{Z}$	17	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$	1

Thank you.