ARTIN L-FUNCTIONS OF SMALL CONDUCTOR

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Abstract. We study the problem of finding the Artin L-functions with the smallest conductor for a given Galois type. We adapt standard analytic techniques to our novel situation of fixed Galois type and get much improved lower bounds on the smallest conductor. For small Galois types we use complete tables of number fields to determine the actual smallest conductor.

Contents

1. Overview

Artin L-functions $L(\mathcal{X}, s)$ are remarkable analytic objects built from number fields. Let $\overline{\mathbf{Q}}$ be the algebraic closure of the rational number field $\mathbf{Q}$ inside the field of complex numbers $\mathbf{C}$. Then Artin L-functions are indexed by continuous complex characters $\mathcal{X}$ of the absolute Galois group $\mathbb{G} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, with the unital character 1 giving the Riemann zeta function $L(1, s) = \zeta(s)$. An important problem in modern number theory is to obtain a fuller understanding of these higher analogs of the Riemann zeta function. The analogy is expected to be very tight: all Artin L-functions are expected by the Artin conjecture to be entire except perhaps for a pole at $s = 1$; they are all expected to satisfy the Riemann hypothesis that all zeros with $\mathrm{Re}(s) \in (0, 1)$ satisfy $\mathrm{Re}(s) = 1/2$.

The two most basic invariants of an Artin L-function $L(\mathcal{X}, s)$ are defined via the two explicit elements of $\mathbb{G}$, the identity e and the complex conjugation element σ. These invariants are the degree $n = \mathcal{X}(e)$ and the signature $r = \mathcal{X}(\sigma)$ respectively. A measure of the complexity of $L(\mathcal{X}, s)$ is its conductor $D \in \mathbf{Z}_{\geq 1}$, which can be

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computed from the discriminants of related number fields. It is best for purposes such as ours to focus instead on the root conductor $\delta = D^{1/n}$.

In this paper, we aim to find the simplest Artin L-functions exhibiting a given Galois-theoretic behavior. To be more precise, consider triples (G, c, χ) consisting of a finite group G, an involution $c \in G$, and a faithful character χ. We say that $\mathcal{X}$ has *Galois type* (G, c, χ) if there is a surjection $h : \mathbb{G} \to G$ with $h(\sigma) = c$, and $\mathcal{X} = \chi \circ h$. Let $\mathcal{L}(G, c, \chi)$ be the set of L-functions of type (G, c, χ), and let $\mathcal{L}(G, c, \chi; B)$ be the subset consisting of L-functions with root conductor at most B. Two natural problems for any given Galois type (G, c, χ) are

1: Use known and the above conjectured properties of L-functions to get a lower bound $\mathfrak{d}(G, c, \chi)$ on the root conductors of L-functions in $\mathcal{L}(G, c, \chi)$.
2: Explicitly identify the sets $\mathcal{L}(G, c, \chi; B)$ with B as large as possible.

This paper gives answers to both problems, although for brevity we often fix only (G, χ) and work instead with the sets $\mathcal{L}(G, \chi; B) := \cup_c \mathcal{L}(G, c, \chi; B)$.

There is a large literature on a special case of the situation we study. Namely let (G, c, ϕ) be a Galois type where ϕ is the character of a transitive permutation representation of G. Then the set $\mathcal{L}(G, c, \phi; B)$ is exactly the set of Dedekind zeta functions $\zeta(K, s)$ arising from a corresponding set $\mathcal{K}(G, c, \phi; B)$ of arithmetic equivalence classes of number fields. In this context, root conductors are just root discriminants, and lower bounds date back to Minkowski's work on the geometry of numbers. Use of Dedekind zeta functions as in **1** above began with work of Odlyzko [Odl76, Odl75, Odl77a], Serre [Ser86], Poitou [Poi77a, Poi77b], and Martinet [Mar82]. Extensive responses to **2** came shortly thereafter, with papers often focusing on a single degree $n = \phi(e)$. Early work for quartics, quintics, sextics, and septics include respectively [BF89, For91, BFP93], [SPDyD94], [Poh82, BMO90, Oli91, Oli92, Oli90], and [Lét95]. Further results towards **2** in higher degrees are extractable from the websites associated to [JR14a], [KM01], and [LMF].

The full situation that we are studying here was identified clearly by Odlyzko in [Odl77b], who responded to **1** with a general lower bound. However this more general case of Artin L-functions has almost no subsequent presence in the literature. A noticeable exception is a recent paper of Pizarro-Madariaga [PM11], who improved on Odlyzko's results on **1**. A novelty of our paper is the separation into Galois types. For many Galois types this separation allows us to go considerably further on **1**. This paper is also the first systematic study of **2** beyond the case of number fields.

Sections 2 and 3 review background on Artin L-functions and tools used to bound their conductors. Sections 4–6 form the new material on the lower bound problem **1**, while Sections 7–9 focus on the tabulation problem **2**. Finally, Section 10 returns to **1** and considers asymptotic lower bounds for root conductors of Artin L-functions in certain families. In regard to **1**, Figure 8.1 and Corollary 10.1 give a quick indication of how our type-based lower bounds compare with the earlier degree-based lower bounds. In regard to both **1** and **2**, Tables 8.1–8.8 show how the new lower bounds compare with actual first conductors for many types.

Artin L-functions have recently become much more computationally accessible through a package implemented in *Magma* by Tim Dokchitser. Thousands are now collected in a section on the LMFDB [LMF]. The present work increases our understanding of all this information in several ways, including by providing completeness certificates for certain ranges.

2. Artin L-functions

In this section we provide some background. An important point is that our problems allow us to restrict consideration to Artin characters which take rational values only. In this setting, Artin L-functions can be expressed as products and quotients of roots of Dedekind zeta functions, minimizing the background needed. General references on Artin L-functions include [Mar77, RM01].

2.1. Number fields.

A number field K has many invariants relevant for our study. First of all, there is the degree $n = [K : \mathbf{Q}]$. The other invariants we need are local in that they are associated with a place v of $\mathbf{Q}$ and can be read off from the corresponding completed algebra $K_v = K \otimes \mathbf{Q}_v$, but not from other completions. For $v = \infty$, the complete invariant is the signature r, defined by $K_\infty \cong \mathbf{R}^r \times \mathbf{C}^{(n-r)/2}$. It is more convenient sometimes to work with the eigenspace dimensions for complex conjugation, $a = (n+r)/2$ and $b = (n-r)/2$. For an ultrametric place $v = p$, the full list of invariants is complicated. The most basic one is the positive integer $D_p = p^{c_p}$ generating the discriminant ideal of $K_p/\mathbf{Q}_p$. We package the D_p into the single invariant $D = \prod_p D_p \in \mathbf{Z}_{\geq 1}$, the absolute discriminant of K.

2.2. Dedekind zeta functions.

Associated with a number field is its Dedekind zeta function

$$\zeta(K, s) = \prod_p \frac{1}{P_p(p^{-s})} = \sum_{m=1}^{\infty} \frac{a_m}{m^s}. \tag{2.1}$$

Here the polynomial $P_p(x) \in \mathbf{Z}[x]$ is a p-adic invariant. It has degree $\leq n$ with equality if and only if $D_p = 1$. The integer a_m is the number of ideals of index m in the ring of integers $\mathcal{O}_K$.

2.3. Analytic properties of Dedekind zeta functions.

Let $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, where $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx$ is the standard gamma function. Let

$$\widehat{\zeta}(K, s) = D^{s/2}\Gamma_{\mathbf{R}}(s)^a\,\Gamma_{\mathbf{R}}(s+1)^b\,\zeta(K, s). \tag{2.2}$$

Then this completed Dedekind zeta function $\widehat{\zeta}(K, s)$ meromorphically continues to the whole complex plane, with simple poles at $s = 0$ and $s = 1$ being its only singularities. It satisfies the functional equation $\widehat{\zeta}(K, s) = \widehat{\zeta}(K, 1-s)$.

2.4. Permutation characters.

We recall from the introduction that throughout this paper we are taking $\overline{\mathbf{Q}}$ to be the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$ and $\mathbb{G} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ its absolute Galois group. A degree n number field K then corresponds to the transitive n-element $\mathbb{G}$-set $X = \mathrm{Hom}(K, \overline{\mathbf{Q}})$. A number field thus has a permutation character $\Phi = \Phi_K = \Phi_X$ with $\Phi(e) = n$. Also signature has the character-theoretic interpretation $\Phi(\sigma) = r$, where σ as before is the complex conjugation element.

2.5. **General characters and Artin L-functions.** Let $\mathcal{X}$ be a character of $\mathbb{G}$. Then one has an associated Artin L-function $L(\mathcal{X}, s)$ and conductor $D_{\mathcal{X}}$, agreeing with the Dedekind zeta function $\zeta(K, s)$ and the discriminant D_K if $\mathcal{X}$ is the permutation character of K. The function $L(\mathcal{X}, s)$ has both an Euler product and Dirichlet series representation as in (2.1). In general, if $\Phi = \sum_{\mathcal{X}} m_{\mathcal{X}} \mathcal{X}$ then

$$L(\Phi, s) = \prod_{\mathcal{X}} L(\mathcal{X}, s)^{m_{\mathcal{X}}} \qquad\qquad D_{\Phi} = \prod_{\mathcal{X}} D_{\mathcal{X}}^{m_{\mathcal{X}}}. \tag{2.3}$$

One is often interested in (2.3) where the $\mathcal{X}$ are irreducible characters.

For a finite set of primes S, let $\overline{\mathbf{Q}}_S$ be the compositum of all number fields in $\overline{\mathbf{Q}}$ with discriminant divisible only by primes in S. Let $\mathbb{G}_S = \mathrm{Gal}(\overline{\mathbf{Q}}_S/\mathbf{Q})$ be the corresponding quotient of $\mathbb{G}$. Then for primes $p \notin S$ one has a well-defined Frobenius conjugacy class Fr_p in $\mathbb{G}_S$. The local factor $P_p(x)$ in (2.1) is the characteristic polynomial $\det(1 - \rho(\mathrm{Fr}_p)x)$, where ρ is a representation with character $\mathcal{X}$.

2.6. **Relations with other objects.** Artin L-functions of degree 1 are exactly Dirichlet L-functions, so that $\mathcal{X}$ can be identified with a faithful character of the quotient group $(\mathbf{Z}/D\mathbf{Z})^\times$ of $\mathbb{G}$, with D the conductor of $\mathcal{X}$. Artin L-functions coming from irreducible degree 2 characters and conductor D are expected to come from cuspidal modular forms on $\Gamma_1(D)$, holomorphic if $r = 0$ and nonholomorphic otherwise. This expectation is proved in all cases, except for those with $r = \pm 2$ and projective image the nonsolvable group A_5. In general, to understand how an Artin L-function $L(\mathcal{X}, s)$ qualitatively relates to other objects, one needs to understand its Galois theory, including the placement of complex conjugation; in other words, one needs to identify its Galois type. To be more quantitative, one brings in the conductor.

2.7. **Analytic Properties of Artin L-functions.** An Artin L-function has a meromorphic continuation and functional equation, although each with an extra complication in comparison with the special case of Dedekind zeta functions. For the meromorphic continuation, the behavior at $s = 1$ is known: the pole order is the multiplicity $(1, \mathcal{X})$ of 1 in $\mathcal{X}$. The complication is that one has poor control over other possible poles. The Artin conjecture for $\mathcal{X}$ says however that there are no poles other than $s = 1$.

The completed L-function

$$\widehat{L}(\mathcal{X}, s) = D_{\mathcal{X}}^{s/2} \Gamma_{\mathbf{R}}(s)^a \Gamma_{\mathbf{R}}(s+1)^b L(\mathcal{X}, s)$$

satisfies the functional equation

$$\widehat{L}(\mathcal{X}, 1-s) = w \widehat{L}(\overline{\mathcal{X}}, s)$$

with root number w. Irreducible characters of any compact group come in three types, orthogonal, non-real, and symplectic. The type is identified by the Frobenius-Schur indicator, calculated with respect to the Haar probability measure dg:

$$FS(\chi) = \int_G \chi(g^2)\, dg \in \{-1, 0, 1\}.$$

For orthogonal characters $\mathcal{X}$ of $\mathbb{G}$, one has $\mathcal{X} = \overline{\mathcal{X}}$ and moreover $w = 1$. The complication in comparison with permutation characters is that for the other two types, the root number w is not necessarily 1. For symplectic characters, $\mathcal{X} = \overline{\mathcal{X}}$

and w can be either of the two possibilities 1 or -1. For non-real characters, $\mathcal{X} \neq \overline{\mathcal{X}}$ and w is some algebraic number of norm 1.

Recall from the introduction that an Artin L-function is said to satisfy the Riemann hypothesis if all its zeros in the critical strip $0 < \mathrm{Re}(s) < 1$ are actually on the critical line $\mathrm{Re}(s) = 1/2$. We will be using the Riemann hypothesis through Lemma 3.1. If we replaced the function (3.1) with the appropriately scaled version of (5.17) from [PM11], then our lower bounds would be only conditional on the Artin conjecture, which is completely known for some Galois types (G, c, χ). However the bounds obtained would be much smaller, and the comparison with first conductors as presented in Tables 8.1–8.8 below would be less interesting.

2.8. **Rational characters and rational Artin L-functions.** The abelianized Galois group $\mathbb{G}^{\mathrm{ab}}$ acts on continuous complex characters of profinite groups through its action on their values. If $\mathcal{X}'$ and $\mathcal{X}''$ are conjugate via this action then their conductors agree:

$$D_{\mathcal{X}'} = D_{\mathcal{X}''}. \tag{2.4}$$

Our study is simplified by this equality because it allows us to study a given irreducible character $\mathcal{X}'$ by studying instead the rational character $\mathcal{X}$ obtained by summing its conjugates.

By the Artin induction theorem [Fei67, Prop. 13.2], a rational character $\mathcal{X}$ can be expressed as a rational linear combination of permutation characters:

$$\mathcal{X} = \sum k_\Phi \Phi. \tag{2.5}$$

For general characters $\mathcal{X}'$, computing the Frobenius traces $a_p = \mathcal{X}'(\mathrm{Fr}_p)$ requires the results of [DD13]. Similarly the computation of bad Euler factors and the root number w present difficulties. For Frobenius traces and bad Euler factors, these complications are not present for rational characters $\mathcal{X}$ because of (2.5).

3. Signature-based analytic lower bounds

Here and in the next section we aim to be brief, with the main point being to explain how type-based lower bounds are usually much larger than signature-based lower bounds. We employ the standard framework for establishing lower bounds for conductors and discriminants, namely Weil's explicit formula. General references for the material in this section are [Odl77b, PM11].

3.1. **Basic quantities.** The theory allows general test functions that satisfy certain axioms. We work only with a function introduced by Odlyzko (see [Poi77b, (9)]),

$$f(x) = \begin{cases} (1-x)\cos(\pi x) + \dfrac{\sin(\pi x)}{\pi}, & \text{if } 0 \le x \le 1, \\ 0, & \text{if } x > 1. \end{cases} \tag{3.1}$$

For $z \in [0, \infty)$ let

$$\begin{aligned} N(z) &= \log(\pi) + \int_0^\infty \frac{e^{-x/4} + e^{-3x/4}}{2(1-e^{-x})} f(x/(2z)) - \frac{e^{-x}}{x}\, dx, \\ &= \gamma + \log(8\pi) + \int_0^\infty \frac{f(x/z) - 1}{2\sinh(x/2)}\, dx \end{aligned}$$

$$\begin{aligned}
&= \gamma + \log(8\pi) + \int_0^z \frac{f(x/z)-1}{2\sinh(x/2)}\,dx - \log\left(\frac{e^{z/2}+1}{e^{z/2}-1}\right),\\
R(z) &= \int_0^\infty \frac{e^{-x/4}-e^{-3x/4}}{2(1-e^{-x})} f(x/(2z))\,dx,\\
&= \int_0^z \frac{f(x/z)}{2\cosh(x/2)}\,dx,\\
P(z) &= 4\int_0^\infty f(x/z)\cosh(x/2)\,dx\\
&= \frac{256\pi^2 z\cosh^2(z/4)}{(z^2+4\pi^2)^2}.
\end{aligned}$$

The simplifications in the integrals for $N(z)$ and $R(z)$ are fairly standard and apply to any test function, with the exception of the final steps which make use of the support for $f(x)$. Evaluation of $P(z)$ depends on the choice of $f(x)$. The integrals for $N(z)$ and $R(z)$ cannot be evaluated in closed form like the third, but, as indicated in [Poi77b, §2], they do have simple limits $N(\infty) = \log(8\pi) + \gamma$ and $R(\infty) = \pi/2$ as $z \to \infty$. Here $\gamma \approx 0.5772$ is the Euler γ-constant. The constants $\Omega = e^{N(\infty)} \approx 44.7632$ and $e^{R(\infty)} \approx 4.8105$, as well as their product $\Theta = e^{N(\infty)+R(\infty)} \approx 215.3325$, will play important roles in the sequel.

3.2. **The quantity** $M(n,r,u)$**.** Consider triples (n,r,u) of real numbers with n and u positive and $r \in [-n,n]$. For such a triple, define

$$M(n,r,u) = \mathrm{Max}_z\left(\exp\left(N(z) + \frac{r}{n}R(z) - \frac{u}{n}P(z)\right)\right).$$

It is clear that $M(n,r,u) = M(n/u, r/u, 1)$. Accordingly we regard $u=1$ as the essential case and abbreviate $M(n,r) = M(n,r,1)$. For fixed $\epsilon \in [0,1]$ and $u > 0$, one has the asymptotic behavior

$$\lim_{n\to\infty} M(n,\epsilon n) = \Omega^{1-\epsilon}\Theta^{\epsilon} \approx 44.7632^{1-\epsilon}215.3325^{\epsilon}. \tag{3.2}$$

Figure 3.1 gives one a feel for the fundamental function $M(n,r)$. Particularly important are the univariate functions $M(n,0)$ and $M(n,n)$ corresponding to the left and right edges of this figure.

3.3. **Lower bounds for root discriminants.** Suppose that Φ is a nonzero Artin character which takes real values only. We say that Φ is nonnegative if

$$\Phi(g) \geq 0 \text{ for all } g \in \mathbb{G}. \tag{3.3}$$

This nonnegativity ensures that the inner product $(\Phi, 1)$ of Φ with the unital character 1 is positive. A central result of the theory, a special case of the statement in [Poi77b, (7)], then serves us as a lemma.

Lemma 3.1. *The lower bound*

$$\delta_\Phi \geq M(n,r,u).$$

is valid for all nonnegative characters Φ *with* $(\Phi(e), \Phi(\sigma), (\Phi,1)) = (n,r,u)$ *and* $L(\Phi,s)$ *satisfying the Artin conjecture and the Riemann hypothesis.*

If Φ is a permutation character, then the nonnegativity condition (3.3) is automatically satisfied. This makes the application of the analytic theory to lower bounds of root discriminants of fields relatively straightforward.

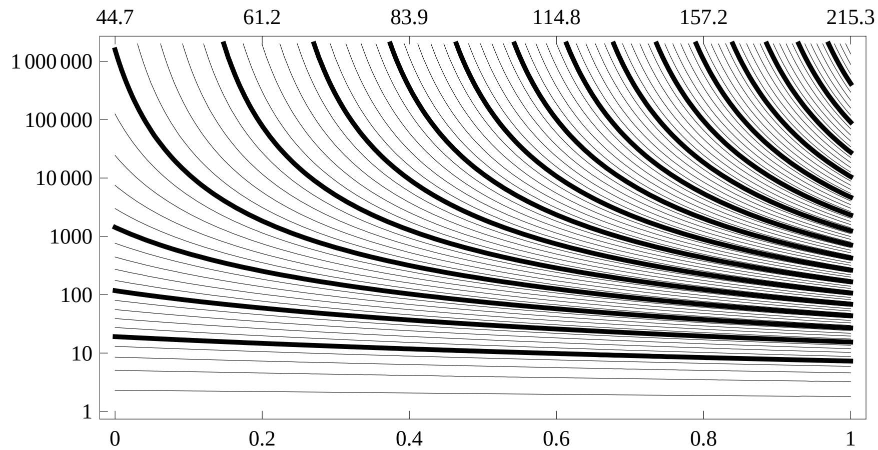

FIGURE 3.1. A contour plot of $M(n, \epsilon n)$ in the window $[0, 1] \times [1, 1\,000\,000]$ of the ϵ-n plane, with a vertical logarithmic scale and contours at 2, 4, 6, 8, **10**, ..., **170**, 172, 174, 176. Some limits for $n \to \infty$ are shown on the upper boundary.

3.4. **Lower bounds for general Artin conductors.** To pass from nonnegative characters to general characters, the classical method uses the following lemma.

Lemma 3.2 (Odlyzko [Odl77b]). *The conductor relation*

$$\delta_{\mathcal{X}} \geq \delta_{\Phi}^{n/(2n-2)} \tag{3.4}$$

holds for any degree n character $\mathcal{X}$ and its absolute square $\Phi = |\mathcal{X}|^2$.

A proof of this lemma from first principles is given in [Odl77b].

Combining Lemma 3.1 with Lemma 3.2 one gets the following immediate consequence

Theorem 3.3. *The lower bound*

$$\delta_{\mathcal{X}} \geq M(n^2, r^2, w)^{n/(2n-2)}$$

is valid for all characters $\mathcal{X}$ with $(\mathcal{X}(e), \mathcal{X}(\sigma), (\mathcal{X}, \overline{\mathcal{X}})) = (n, r, w)$ such that $L(|\mathcal{X}|^2, s)$ satisfies the Artin conjecture and the Riemann hypothesis.

This theorem is essentially the main result in the literature on lower bounds for Artin conductors. It appears in [Odl77b, PM11] with the right side replaced by explicit bounds. For fixed $\epsilon \in [-1, 1]$ and $w > 0$, one has the asymptotic behavior

$$\lim_{n\to\infty} M(n^2, \epsilon^2 n^2, w) = \Omega^{(1-\epsilon^2)/2}\Theta^{\epsilon^2/2} \approx 6.6905^{1-\epsilon^2} 14.6742^{\epsilon^2}. \tag{3.5}$$

The bases $\Omega \approx 44.7632$ and $\Theta \approx 215.3325$ of (3.2) serve as limiting lower bounds for root discriminants via Lemma 3.1. However it is only their square roots $\sqrt{\Omega} \approx 6.6905$ and $\sqrt{\Theta} \approx 14.6742$ which Theorem 3.3 gives as limiting lower bounds for root conductors. This discrepancy will be addressed in Section 10.

4. Type-based analytic lower bounds

In this section we establish Theorem 4.2, which is a family of lower bounds on the root conductor $\delta_{\mathcal{X}}$ of a given Artin character, dependent on the choice of an auxiliary character ϕ.

4.1. **Conductor relations.** Let G be a finite group, c an involution in G, χ a faithful character of G, and ϕ a non-zero real-valued character of G. Say that a pair of Artin characters ($\mathcal{X}$,Φ) has joint type (G, c, χ, ϕ) if there is a surjection $h : \mathbb{G} \to G$ with $h(\sigma) = c$, $\mathcal{X} = \chi \circ h$, and $\Phi = \phi \circ h$.

Write the conductors respectively as

$$D_{\mathcal{X}} = \prod_p p^{c_p(\mathcal{X})}, \qquad D_\Phi = \prod_p p^{c_p(\Phi)}.$$

Just as in the last section, we need lower bounds on $D_{\mathcal{X}}$ in terms of D_Φ. Our paper [JR14b] produces bounds of this sort in the context of many characters. Here we present some of these results restricted to the setting of two characters, but otherwise following the notation of [JR14b].

For $\tau \in G$, let $\bar{\tau}$ be its order. Let ψ be a rational character of G. Define two similar numbers,

$$\widehat{c}_\tau(\psi) = \psi(e) - \psi(\tau), \qquad c_\tau(\psi) = \psi(e) - \frac{1}{\bar{\tau}} \sum_{k|\bar{\tau}} \varphi(\bar{\tau}/k)\psi(\tau^k). \tag{4.1}$$

Here φ is the Euler totient function given by $\varphi(k) = |(\mathbf{Z}/k)^\times|$. For the identity element e, one clearly has $\widehat{c}_e(\psi) = c_e(\psi) = 0$. When $\bar{\tau}$ is prime, the functions on rational characters defined in (4.1) are proportional: $(\bar{\tau} - 1)\widehat{c}_\tau(\psi) = \bar{\tau} c_\tau(\psi)$.

The functions $\widehat{c}_\tau$ and c_τ are related to ramification as follows. Let Ψ be an Artin character corresponding to ψ under h. If Ψ is tame at p then

$$c_p(\Psi) = c_\tau(\psi), \tag{4.2}$$

for τ corresponding to a generator of tame inertia. The identity (4.2) holds because $c_\tau(\psi)$ is the number of non-unital eigenvalues of $\rho(\tau)$ for a representation ρ with character ψ. For general Ψ, there is a canonical expansion

$$c_p(\Psi) = \sum_\tau k_\tau \widehat{c}_\tau(\psi), \tag{4.3}$$

with always $k_\tau \geq 0$, coming from the filtration by higher ramification groups on the p-adic inertial subgroup of G.

Because (4.1)–(4.3) are only correct for ψ rational, when we apply them to characters χ and ϕ of interest, we are always assuming that χ and ϕ are rational. As explained in §2.8, the restriction to rational characters still allows obtaining general lower bounds. Also, as will be illustrated by an example in §5.5, focusing on rational characters does not reduce the quality of these lower bounds.

For the lower bounds we need, we define the parallel quantities

$$\widehat{\alpha}(G, \chi, \phi) = \min_{\tau \in G - \{e\}} \frac{\widehat{c}_\tau(\chi)}{\widehat{c}_\tau(\phi)}, \qquad \alpha(G, \chi, \phi) = \min_{\tau \in G - \{e\}} \frac{c_\tau(\chi)}{c_\tau(\phi)}. \tag{4.4}$$

Let $B(G, \chi, \phi)$ be the best lower bound, valid for all primes p, that one can make on $c_p(\mathcal{X})/c_p(\Phi)$ by purely local arguments. As emphasized in [JR14b, §2], $B(G, \chi, \phi)$

can in principle be calculated by individually inspecting all possible p-adic ramification behaviors. The above discussion says

$$\widehat{\alpha}(G,\chi,\phi) \leq B(G,\chi,\phi) \leq \alpha(G,\chi,\phi). \tag{4.5}$$

The left inequality holds because of the nonnegativity of the k_τ in (4.3). The right inequality holds because of (4.2).

A central theme of [JR14b] is that one is often but not always in the extreme situation

$$B(G,\chi,\phi) = \alpha(G,\chi,\phi). \tag{4.6}$$

For example, it often occurs in practice that the minimum in the expression (4.4) for $\widehat{\alpha}(G,\chi,\phi)$ occurs at a τ of prime order. Then the proportionality remark above shows that in fact all three quantities in (4.5) are the same, and so in particular (4.6) holds. As a quite different example, Theorem 7.3 of [JR14b] says that if ϕ is the regular character of G and χ is a permutation character, then (4.6) holds. Some other examples of (4.6) are worked out in [JR14b] by explicit analysis of wild ramification; a few examples show that $B(G,\chi,\phi) < \alpha(G,\chi,\phi)$ is possible too.

4.2. **Root conductor relations.** To switch the focus from conductors to root conductors, we multiply all three quantities in (4.5) by $\phi(e)/\chi(e)$ to obtain

$$\underline{\widehat{\alpha}}(G,\chi,\phi) \leq b(G,\chi,\phi) \leq \underline{\alpha}(G,\chi,\phi). \tag{4.7}$$

Here the elementary purely group-theoretic quantity $\underline{\widehat{\alpha}}(G,\chi,\phi)$ is improved to the best bound $b(G,\chi,\phi)$ which in turn often agrees with a second more complicated but still purely group-theoretic quantity $\underline{\alpha}(G,\chi,\phi)$. The notations $\widehat{\alpha}$, α, $\underline{\widehat{\alpha}}$, $\underline{\alpha}$ are all taken from Section 7 of [JR14b] while the notations B and b correspond to quantities not named in [JR14b].

Our discussion establishes the following lemma.

Lemma 4.1. *The conductor relation*

$$\delta_{\mathcal{X}} \geq \delta_{\Phi}^{b(G,\chi,\phi)} \tag{4.8}$$

holds for all pairs of Artin characters $(\mathcal{X},\Phi)$ with joint type of the form (G,c,χ,ϕ).

4.3. **Bounds via an auxiliary Artin character Φ.** For $u \in \{\underline{\widehat{\alpha}}, b, \underline{\alpha}\}$, define

$$m(G,c,\chi,\phi,u) = M(\phi(e),\phi(c),(\phi,1))^{u(G,\chi,\phi)}. \tag{4.9}$$

Just like Lemma 3.1 combined with Lemma 3.2 to give Theorem 3.3, so too Lemma 3.1 combines with Lemma 4.1 to give the following theorem.

Theorem 4.2. *The lower bound*

$$\delta_{\mathcal{X}} \geq m(G,c,\chi,\phi,b) \tag{4.10}$$

is valid for all character pairs $(\mathcal{X},\Phi)$ of joint type (G,c,χ,ϕ) such that Φ is nonnegative and $L(\Phi,s)$ satisfies the Artin conjecture and the Riemann hypothesis.

Computing the right side of (4.10) is difficult because the base in (4.9) requires evaluating the maximum of a complicated function, while the exponent $b(G,\chi,\phi)$ involves an exhaustive study of wild ramification. Almost always in the sequel, χ and ϕ are rational-valued and we replace $b(G,\chi,\phi)$ by $\underline{\widehat{\alpha}}(G,\chi,\phi)$; in the common case that all three quantities of (4.7) are equal, this is no loss.

5. Four choices for ϕ

This section fixes a type (G, c, χ) where the faithful character χ is rational-valued and uses the notation $(n, r) = (\chi(e), \chi(c))$. The section introduces four nonnegative characters ϕ_i built from (G, χ). For the first character ϕ_L, it makes $m(G, c, \chi, \phi_L, b)$, the lower bound appearing in Theorem 4.2, more explicit. For the remaining three characters ϕ_i, it makes the perhaps smaller quantity $m(G, c, \chi, \phi_i, \widehat{\underline{\alpha}})$ more explicit.

Two simple quantities enter into the constructions as follows. Let X be the set of values of χ, so that $X \subset \mathbf{Z}$ by our rationality assumption. Let $-\check{\chi}$ be the least element of X. The greatest element of X is of course $\chi(e) = n$, and we let $\widehat{\chi}$ be the second greatest element. Thus, $-\check{\chi} < 0 \leq \widehat{\chi} \leq n - 1$.

5.1. **Linear auxiliary character.** A simple nonnegative character associated to χ is $\phi_L = \chi + \check{\chi}$. Both $m(G, c, \chi, \phi_L, \widehat{\underline{\alpha}})$ and $m(G, c, \chi, \phi_L, \underline{\alpha})$ easily evaluate to

$$m(G, c, \chi, \phi_L, b) = M(n + \check{\chi}, r + \check{\chi}, \check{\chi})^{(n+\check{\chi})/n}. \tag{5.1}$$

The character ϕ_L seems most promising as an auxiliary character when $\check{\chi}$ is very small.

In [PM11, §3] the auxiliary character $\chi + n$ is used, which has the advantage of being nonnegative for any rational character χ. Odlyzko also uses $\chi + n$ in [Odl77b], and suggests using the auxiliary character $\phi_L = \chi + \check{\chi}$ since it gives a better bound whenever $\check{\chi} < n$. This strict inequality holds exactly when the center of G has odd order.

5.2. **Square auxiliary character.** Another very simple nonnegative character associated to χ is $\phi_S = \chi^2$. This character gives

$$m(G, c, \chi, \phi_S, \widehat{\underline{\alpha}}) = M(n^2, r^2, (\chi, \chi))^{n/(n+\widehat{\chi})}. \tag{5.2}$$

The derivation of (5.2) uses the simple formula in (4.1) for $\widehat{c}_\tau$. The formula for c_τ in (4.1) is more complicated, and we do not expect a simple general formula for $m(G, c, \chi, \phi_S, \underline{\alpha})$, nor for the best bound $m(G, c, \chi, \phi_S, b)$ in Theorem 4.2.

The character ϕ_S is used prominently in [Odl77b, PM11]. When $\widehat{\chi} = n - 2$, the lower bound $m(G, c, \chi, \phi_S, \widehat{\underline{\alpha}})$ coincides with that of Lemma 3.2. Thus for $\widehat{\chi} = n-2$, Theorem 4.2 with $\phi = \phi_S$ gives the same bound as Theorem 3.3. On the other hand, as soon as $\widehat{\chi} < n - 2$, Theorem 4.2 with $\phi = \phi_S$ is stronger. The remaining case $\widehat{\chi} = n-1$ occurs only three times among the 195 characters we consider in Section 8. In these three cases, the bound in Theorem 3.3 is stronger because the exponent is larger. However, in each of these cases, the tame-wild principle applies [JR14b] and we can use exponent $m(G, c, \chi, \phi_S, \underline{\alpha})$, which gives the same bound as Theorem 3.3 in two cases, and a better bound in the third.

5.3. **Quadratic auxiliary character.** Let $-\widetilde{\chi}$ be the greatest negative element of the set X of values of χ. A modification of the given character χ is $\chi^* = \chi + \widetilde{\chi}$, with degree $n^* = n + \widetilde{\chi}$ and signature $r^* = r + \widetilde{\chi}$. A modification of ϕ_S is $\phi_Q = \chi\chi^*$. The function ϕ_Q takes only nonnegative values because the interval $(-\tilde{\chi}, 0)$ in the x-line where $x(x + \tilde{\chi})$ is negative is disjoint from the set X of values of χ. The lower bound associated to ϕ_Q is

$$m(G, c, \chi, \phi_Q, \widehat{\underline{\alpha}}) = M(nn^*, rr^*, (\chi, \chi))^{(n^*)/(n^*+\widehat{\chi})}. \tag{5.3}$$

Comparing formulas (5.2) and (5.3), n^2 strictly increases to nn^* and $n/(n + \widehat{\chi})$ increases to $n^*/(n^* + \widehat{\chi})$. In the totally real case $n = r$, the monotonicity of the function $M(n,n)$ as exhibited in the right edge of Figure 3.1 then implies that $m(G,c,\chi,\phi_S,\underline{\widehat{\alpha}})$ strictly increases to $m(G,c,\chi,\phi_Q,\underline{\widehat{\alpha}})$. Even outside the totally real setting, one can expect that ϕ_Q almost always yields a better lower bound than ϕ_S. The character ϕ_Q seems promising as an auxiliary character when $\widehat{\chi}$ is very small so that the exponent is near 1 rather than its lower limit of 1/2. As for the square case, we do not expect a simple formula for the best bound $m(G,c,\chi,\phi_Q,b)$ in Theorem 4.2.

5.4. **Galois auxiliary character.** Finally there is a strong candidate for a good auxiliary character that does not depend on χ, namely the regular character ϕ_G. By definition, $\phi_G(e) = |G|$ and else $\phi_G(g) = 0$. In this case one has

$$m(G,c,\chi,\phi_G,\underline{\widehat{\alpha}}) = M(|G|,\delta_{ce}|G|,1)^{(n-\widehat{\chi})/n}. \tag{5.4}$$

Here δ_{ce} is defined to be 1 in the totally real case and 0 otherwise. This auxiliary character again seems most promising when $\widehat{\chi}$ is small. As in the square and quadratic cases, we do not expect a simple formula for $m(G,c,\chi,\phi_G,b)$.

5.5. **Spectral bounds and rationality.** To get large lower bounds on root conductors, one wants $\breve{\chi}/n$ to be small for (5.1) or $\widehat{\chi}/n$ to be small for (5.2)–(5.4). The analogous quantities $\breve{\chi}_1/n_1$ and $\widehat{\chi}/n_1$ are well-defined for a general real character χ_1, and replacing χ_1 by the sum χ of its conjugates can substantially reduce them.

For example, let p be a prime congruent to 1 modulo 4, and let G be the simple group $\mathrm{PSL}_2(p)$. Then G has two irrational irreducible characters, say χ_1 and χ_2, both of degree $(p+1)/2$. For each, its set of values is

$$\left\{\frac{-\sqrt{p}-1}{2}, -1, 0, 1, \frac{\sqrt{p}-1}{2}, \frac{p+1}{2}\right\}$$

(except that 1 is missing if $p = 5$). However for $\chi = \chi_1+\chi_2$, the set of values is just $\{-2, 0, 2, p+1\}$. Thus in passing from $\breve{\chi}_1/n_1$ to $\breve{\chi}/n$, one saves a factor of $\sqrt{p}+1$. Similarly in passing from $\widehat{\chi}_1/n_1$ to $\widehat{\chi}/n$, one saves a factor of $\sqrt{p}-1$.

6. Other choices for ϕ

To apply Theorem 4.2 for a given Galois type (G,c,χ), one needs to choose an auxiliary character ϕ. We presented four choices in Section 5. We discuss all possible choices here, using $G = A_4$ and $G = A_5$ as illustrative examples.

6.1. **Rational character tables.** As a preliminary, we review the notion of rational character table. Let $G^\sharp = \{C_j\}_{j\in J}$ be the set of power-conjugacy classes in G. Let $G^{\mathrm{rat}} = \{\chi_i\}_{i\in I}$ be the set of rationally irreducible characters. These sets have the same size k and one has a $k\times k$ matrix $\chi_i(C_j)$, called the rational character table.

A_4	$1A$	$2A$	$3AB$
χ_1	1	1	1
χ_2	2	2	−1
χ_3	3	−1	0

A_5	$1A$	$2A$	$3A$	$5AB$
χ_1	1	1	1	1
χ_4	4	0	1	−1
χ_5	5	1	−1	0
χ_6	6	−2	0	1

TABLE 6.1. Rational character tables for A_4 and A_5

Two examples are given in Table 6.1. We index characters by their degree, with $I = \{1, 2, 3\}$ for A_4 and $I = \{1, 4, 5, 6\}$ for A_5. All characters are absolutely irreducible except for χ_2 and χ_6, which each break as a sum of two conjugate irreducible complex characters. We likewise index power-conjugacy classes by the order of a representing element, always adding letters as is traditional. Thus $J = \{1A, 2A, 3AB\}$ for A_4 and $J = \{1A, 2A, 3A, 5AB\}$ for A_5, with $3AB$ and $5AB$ each consisting of two conjugacy classes.

6.2. **The polytope P_G of normalized nonnegative characters.** A general real-valued function $\phi \in \mathbf{R}(G^\sharp)$ has an expansion $\sum x_i \chi_i$ with $x_i \in \mathbf{R}$. The coefficients are recovered via inner products, $x_i = (\phi, \chi_i)/(\chi_i, \chi_i)$. Alternative coordinates are given by $y_j = \phi(C_j)$. The ϕ allowed for Theorem 4.2 are the nonzero ϕ with the x_i and y_j non-negative integers.

An allowed ϕ gives the same lower bound in Theorem 4.2 as any of its positive multiples $m\phi$. Without getting any new bounds, we can therefore give ourselves the convenience of allowing the x_i and y_j to be nonnegative rational numbers. Similarly, we can extend by continuity to allow the x_i and y_j to be nonnegative real numbers. The allowed ϕ then become the cone in k-dimensional Euclidean space given by $x_i \geq 0$ and $y_j \geq 0$, excluding the tip of the cone at the origin.

Writing the identity character as χ_1, we can normalize via scaling to $x_1 = 1$. Writing the identity class as C_{1A}, the inequality $y_{1A} \geq 0$ is implied by the other $y_j \geq 0$ and so the variable y_{1A} can be ignored. The polytope P_G of normalized nonnegative characters is then defined by $x_1 = 1$, the inequalities $x_i \geq 0$ for $i \neq 1$, and inequalities $y_j \geq 0$ for $j \neq 1A$. The point where all the x_i are zero is the unital character ϕ_1. The point where all the y_j are zero is the regular character ϕ_G. Thus the $(k-1)$-dimensional polytope P_G is determined by $2k-2$ linear inequalities, with $k-1$ corresponding to non-unital characters and intersecting at ϕ_1, and $k-1$ corresponding to non-identity classes and intersecting at ϕ_G.

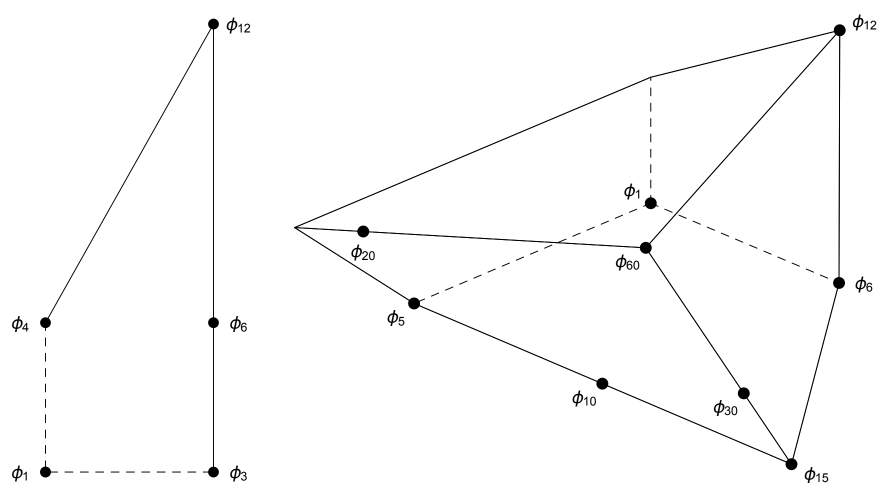

TABLE 6.2. The polytopes P_{A_4} and P_{A_5}

Figure 6.2 continues our two examples. On the left, P_{A_4} is drawn in the x_2-x_3 plane. The character faces give the coordinate axes and are dashed. The class faces are calculated from columns in the rational character table and are solid. On the right, a view of P_{A_5} is given in x_4-x_5-x_6 space. The three pairwise intersections of character faces give coordinate axes and are dashed, while all other edges are solid. In this view, the point $\phi_G = \phi_{60} = (4, 5, 3)$ should be considered as closest to the reader, with the solid lines visible and the dashed lines hidden by the polytope. Note that P_{A_4} has the combinatorics of a square and P_{A_5} has the combinatorics of a cube. While the general P_G is the intersection of an orthant with tip ϕ_1 and an orthant with tip ϕ_G, its combinatorics are typically more complicated than $[0, 1]^{(k-1)}$. For example, the groups $G = A_6$, S_5, A_7, and S_6, have $k = 6$, 7, 8, and 11 respectively; but instead of having 32, 64, 128 and 1024 vertices, their polytopes P_G have 28, 40, 115, and 596 vertices respectively.

6.3. **Points in P_G.** In the previous subsection, we have mentioned already the distinguished vertices ϕ_1 and ϕ_G. For every rationally irreducible character, we also have $\phi_{\chi,L} = \chi + \breve{\chi}$, $\phi_{\chi,S} = \chi^2$, and $\phi_{\chi,Q} = \chi\chi^*$, as in Section 5.

For every subgroup H of G, another element of P_G is the permutation character $\phi_{G/H}$. For $H = G$, this character is just the ϕ_1 considered before, which is a vertex. Otherwise, a theorem of Jordan, discussed at length in [Ser03], says that $\phi_{G/H}(C_j) = 0$ for at least one j; in other words, $\phi_{G/H}$ is on at least one character face. For A_4 and A_5, there are respectively five and nine conjugacy classes of subgroups, distinguished by their orders. Figures 6.2 draws the corresponding points, labeled by $\phi_{|G/H|}$. All four vertices of P_{A_4} and six of the eight vertices of P_{A_5} are of the form ϕ_N. The remaining one ϕ_N in P_{A_4} is on an edge, while the remaining three ϕ_N in P_{A_5} are on edges as well.

6.4. **The best choice for ϕ.** Given (G, c, χ) and $u \in \{\widehat{\alpha}, b, \underline{\alpha}\}$, let $m(G, c, \chi, u) = \max_{\phi \in P_G} m(G, c, \chi, \phi, u)$. Computing these maxima seems difficult. Instead we vary ϕ over a modestly large finite set, denoting the largest bound appearing as $\mathfrak{d}(G, c, \chi, u)$. For most G, the set of ϕ we inspect consists of all $\phi_{\chi,L}$, $\phi_{\chi,S}$, and $\phi_{\chi,Q}$, all $\phi_{G/H}$ including the regular character ϕ_G, and all vertices. For some G, like S_7, there are too many vertices and we exclude them from the list of ϕ we try.

For each (G, χ), we work either with $u = \widehat{\alpha}$ or with $u = \underline{\alpha}$, as explained in the "middle four columns" part of §8.2.2. We then report $\mathfrak{d}(G, \chi) = \min_c \mathfrak{d}(G, c, \chi, u)$ in Section 8.

7. The case $G = S_5$

Our focus in the next two sections is on finding initial segments $\mathcal{L}(G, \chi; B)$ of complete lists of Artin L-functions, and in particular on finding the first root conductor $\delta_1(G, \chi)$. It is a question of transferring completeness statements for number fields to completeness statements for Artin L-functions via conductor relations. In this section, we explain the process by presenting the case $G = S_5$ in some detail.

7.1. **Different orders on the same set of fields.** Consider the set $\mathcal{K}$ of isomorphism classes of quintic fields K over $\mathbf{Q}$ with splitting field $L/\mathbf{Q}$ having Galois group $\mathrm{Gal}(L/\mathbf{Q}) \cong S_5$. The group S_5 has seven irreducible characters which we index by degree and an auxiliary label: $\chi_{1a} = 1$, χ_{1b}, χ_{4a}, χ_{4b}, χ_{5a}, χ_{5b}, and χ_{6a}. For ϕ a permutation character, let $D_\phi(K) = D(K_\phi)$ be the absolute discriminant of the associated resolvent algebra K_ϕ of K. Extending by multiplicativity, functions $D_\chi : \mathcal{K} \to \mathbf{R}_{>0}$ are defined for general $\chi = \sum m_n \chi_n$. They do not depend on the coefficient m_{1a}. We follow our practice of often shifting attention to the corresponding root conductors $\delta_\chi(K) = D_\chi(K)^{1/\chi(e)}$.

λ_5	1^5	2^21	31^2	5	21^3	41	32	1^5	2^21	31^2	5	21^3	41	32		
λ_6	1^6	2^21^2	33	51	2^3	41^2	6	1^6	2^21^2	33	51	2^3	41^2	6	$\widehat{\alpha}(n)$	$\underline{\alpha}(n)$
χ_{1a}	1	1	1	1	1	1	1	0	0	0	0	0	0	0		
χ_{1b}	1	1	1	1	−1	−1	−1	0	0	0	0	1	1	1		
χ_{4a}	4	0	1	−1	**2**	0	−1	0	2	2	4	**1**	3	3	0.50	0.50
χ_{4b}	4	0	**1**	−1	−2	0	**1**	0	2	**2**	4	3	3	3	0.75	0.75
χ_{5a}	5	**1**	−1	0	**1**	−1	**1**	0	**2**	4	4	**2**	4	4	0.80	0.80
χ_{5b}	5	**1**	−1	0	−1	**1**	−1	0	**2**	4	4	3	3	5	0.80	0.80
χ_{6a}	6	−2	0	**1**	0	0	0	0	4	4	**4**	3	5	5	$0.8\overline{3}$	$0.8\overline{3}$
ϕ_{120}	120	0	0	0	0	0	0	0	60	80	96	60	90	100		

TABLE 7.1. Standard character table of S_5 on the left, with entries $\chi_n(\tau)$; tame table [JR14b, §4.3], on the right, with entries $c_\tau(\chi_n)$ as defined in (4.1).

Let $\mathcal{K}(\chi; B) = \{K \in \mathcal{K} : \delta_\chi(K) \leq B\}$. Suppose now all the m_n are nonnegative with at least one coefficient besides m_{1a} and m_{1b} positive. Then δ_χ is a height function in the sense that all the $\mathcal{K}(\chi; B)$ are finite. Suppressing the secondary phenomenon that ties among a finite number of fields can occur, we think of each δ_χ as giving an ordering on the set $\mathcal{K}$.

The orderings coming from different δ_χ can be very different. For example, consider the field $K \in \mathcal{K}$ defined by the polynomial $x^5 - 2x^4 + 4x^3 - 4x^2 + 2x - 4$. This field is the first field in $\mathcal{K}$ when ordered by the regular character $\phi_{120} = \sum_n \chi_n(n)\chi_n$. However it is the 22^{nd} field when ordered by $\phi_6 = 1 + \chi_{5b}$ only the 2298^{th} field when ordered by $\phi_5 = 1 + \chi_{4a}$.

This phenomenon of different orderings on the same set of number fields plays a prominent role in asymptotic studies [Woo10]. Here we are interested instead in initial segments and how they depend on χ. Our formalism lets us treat any χ. Following the conventions for general G of the next section, we focus on the five irreducible χ with $\chi(e) > 1$, thus χ_n for $n \in \{4a, 4b, 5a, 5b, 6a\}$.

7.2. **Computing Artin conductors.** To compute general $D_\chi(K)$, one needs to work with enough resolvents of $K = K_5 = \mathbf{Q}[x]/f_5(x)$. For starters, we have the quadratic resolvent $K_2 = \mathbf{Q}[x]/(x^2 - D(K_5))$ and the Cayley-Weber resolvent $K_6 = \mathbf{Q}[x]/f_6(x)$ [JLY02, JR14b]. The other resolvents we will need are $K_{10} = K_5 \otimes K_2$, $K_{12} = K_2 \otimes K_6$, and $K_{30} = K_5 \otimes K_6$. Defining polynomials are obtained for $K_a \otimes K_b$ by the general formula

$$f_{ab}(x) = \prod_{i=1}^{a} \prod_{j=1}^{b} (x - \alpha_i - \beta_j),$$

where $f_a(x)$ has roots α_i and $f_b(x)$ has roots β_j. So discriminants D_2, D_5, D_6, D_{10}, D_{12}, D_{30} are easily computed.

From the character table, the permutation characters ϕ_N in question are expressed in the basis χ_n as on the left in the following display. Inverting, one gets the χ_n in terms of the ϕ_N as on the right.

$$
\begin{array}{llll}
\phi_2 & = 1 + \chi_{1b}, & \chi_{1b} = -1 + \phi_2, \\
\phi_5 & = 1 + \chi_{4a}, & \chi_{4a} = -1 + \phi_5, \\
\phi_6 & = 1 + \chi_{5b}, & \chi_{4a} = 1 - \phi_2 - \phi_5 + \phi_{10}, \\
\phi_{10} = \phi_5\phi_2 & = 1 + \chi_{1b} + \chi_{4a} + \chi_{4b}, & \chi_{5a} = 1 - \phi_2 - \phi_6 + \phi_{12}, \\
\phi_{12} = \phi_6\phi_2 & = 1 + \chi_{1b} + \chi_{5a} + \chi_{5b}, & \chi_{5b} = -1 + \phi_6, \\
\phi_{30} = \phi_5\phi_6 & = 1 + 2\chi_{4a} + 2\chi_{5a} + \chi_{5b} + \chi_{6a}, & \chi_{6a} = 2\phi_2 - 2\phi_5 + \phi_6 - 2\phi_{12} + \phi_{30}.
\end{array}
$$

Conductors D_n belonging to the χ_n are calculable through these formulas, as e.g. $D_{6a} = D_2^2 D_5^{-2} D_6 D_{12}^{-2} D_{30}$.

For all the groups G considered in the next section, we proceeded similarly. Thus we started with rational character tables from *Magma*. We used linear algebra to express rationally irreducible characters in terms of permutation characters. We used *Magma* again to compute resolvents and then *Pari* to evaluate their discriminants. In this last step, we often confronted large degree polynomials with large coefficients. The discriminant computation was only feasible because we knew *a priori* the set of primes dividing the discriminant, and could then easily compute the p-parts of the discriminants of these resolvent fields for relevant primes p using *Pari/gp* without fully factoring the discriminants of the resolvent polynomials.

Magma's Artin representation package computes conductors of Artin representations in a different and more local manner. Presently, it does not compute all conductors in our range because some decomposition groups are too large.

7.3. **Transferring completeness results.** As an initial complete list of fields, we take $\mathcal{K}(\phi; 85)$ with $\phi = \phi_G = \phi_{120}$. We know from [JR14a] that this set consists of 2080 fields. We list these fields by increasing discriminant, $K^1, \ldots, K^{2080}$, with the resolution of ties conveniently not affecting the explicit results appearing in Table 8.1.

The quantities of Section 4 reappear here, and we will use the abbreviations $\underline{\widehat{\alpha}}(n) = \underline{\widehat{\alpha}}(S_5, \chi_n, \phi)$ and $\underline{\alpha}(n) = \underline{\alpha}(S_5, \chi_n, \phi)$. Since ϕ is zero outside of the identity class, the formulas simplify substantially:

$$
\begin{aligned}
\underline{\widehat{\alpha}}(n) &= \frac{\phi(e)}{\chi_n(e)} \min_\tau \frac{\chi_n(e) - \chi_n(\tau)}{\phi(e) - \phi(\tau)} = 1 - \max_\tau \frac{\chi_n(\tau)}{n}, \\
\underline{\alpha}(n) &= \frac{\phi(e)}{\chi_n(e)} \min_\tau \frac{c_\tau(\chi_n)}{c_\tau(\phi)} = \frac{1}{n} \min_\tau \frac{c_\tau(\chi_n)\overline{\tau}}{\overline{\tau} - 1}.
\end{aligned}
$$

For each of the five n, the classes contributing to the minima are in bold on Table 7.1. So, extremely simply, for computing $\underline{\widehat{\alpha}}(n)$ on the left, the largest $\chi_n(\tau)$ besides $\chi_n(e)$ are in bold. For computing $\underline{\alpha}(n)$ on the right, the $c_\tau(\chi_n)$ with $c_\tau(\chi_n)/c_\tau(\phi)$ minimized are put in bold. For the group S_5, one has agreement $\underline{\widehat{\alpha}}(n) = \underline{\alpha}(n)$ in all five cases. This equality occurs for 170 of the lines in Tables 8.1–8.8, with the other possibility $\underline{\widehat{\alpha}}(n) < \underline{\alpha}(n)$ occurring for the remaining 25 lines.

For any cutoff B, conductor relations give

$$\mathcal{K}(\chi_n; B^{\underline{\widehat{\alpha}}(n)}) \subseteq \mathcal{K}(\phi; B).$$

One has an analogous inclusion for general (G, χ), with ϕ again the regular character for G. When G satisfies the tame-wild principle of [JR14b], the $\underline{\widehat{\alpha}}$ in exponents can be replaced by $\underline{\alpha}$. The group S_5 does satisfy the tame-wild principle, but in this case the replacement has no effect.

The final results are on Table 8.1. In particular for $n = 4a$, $4b$, $5a$, $5b$, $6a$ the unique minimizing fields are K^{103}, K^{21}, K^{14}, K^{6}, and K^{12}, with root conductors approximately 6.33, 18.72, 17.78, 16.27, and 18.18. The lengths of the initial segments identified are 45, 15, 186, 592, and 110. Note that because of the relations $\phi_5 = 1 + \chi_{4a}$ and $\phi_6 = 1 + \chi_{5b}$, the results for $4a$ and $5b$ are just translation of known minima of discriminants of number fields with Galois groups $5T5$ and $6T14$ respectively. For $4b$, $5b$, $6a$, and the majority of the characters on the tables of the next section, the first root conductor and the entire initial segment are new.

8. Tables for 84 groups G

In this section, we present our computational results for small Galois types. For simplicity, we focus on results coming from complete lists of Galois number fields. Summarizing statements are given in §8.1 and then many more details in §8.2.

8.1. **Lower bounds and initial segments.** We consider all groups with a faithful transitive permutation representation in some degree from two to nine, except we exclude the nonsolvable groups in degrees eight and nine. There are 84 such groups, and we consider all associated Galois types (G, χ) with χ a rationally irreducible faithful character. Our first result gives conditional lower bounds:

Theorem 8.1. *For each of the* 195 *Galois types* (G, χ) *listed in Tables 8.1–8.8, the listed value* $\mathfrak{d}$ *gives a lower bound for the root conductor of all Artin representations of type* (G, χ)*, assuming the Artin conjecture and Riemann hypothesis for relevant* L*-functions.*

The bounds in Tables 8.1–8.8 are graphed with the best previously known bounds from [PM11] in Figure 8.1. The horizontal axis represents the dimension $n_1 = \chi_1(e)$ of any irreducible constituent χ_1 of χ. The vertical axis corresponds to lower bounds on root conductors. The piecewise-linear curve connects bounds from [PM11], and there is one dot at height $\mathfrak{d}(G, \chi)$ for each (G, χ) from Tables 8.1–8.8 with $\chi_1(e) \leq 20$. Here we are freely passing back and forth between a rational character χ and an irreducible constituent χ_1 via $\delta_1(G, \chi) = \delta_1(G, \chi_1)$, which is a direct consequence of (2.4).

Not surprisingly, the type-based bounds are larger. In low dimensions n_1, some type-based bounds are close to the general bounds, but by dimension 5 there is a clear separation which widens as the dimension grows. This may in part be explained by the fact that we are only seeing a small number of representations for each of these dimensions. However, as we explain in §10.3, we also expect that the asymptotic lower bound of $\sqrt{\Omega} \approx 6.7$ [PM11] is not optimal, and that this bound is more likely to be at least $\Omega \approx 44.8$.

Our second result unconditionally identifies initial segments:

Theorem 8.2. *For* 144 *Galois types* (G, χ)*, Tables 8.1–8.8 identify a non-empty initial segment* $\mathcal{L}(G, \chi; B^{\beta})$*, and in particular identify the minimal root conductor* $\delta_1(G, \chi)$*.*

8.2. **Tables detailing results on lower bounds and initial segments.** Our tables are organized by the standard doubly-indexed lists of transitive permutation groups mTj, with degrees m running from 2 through 9. Within a degree, the

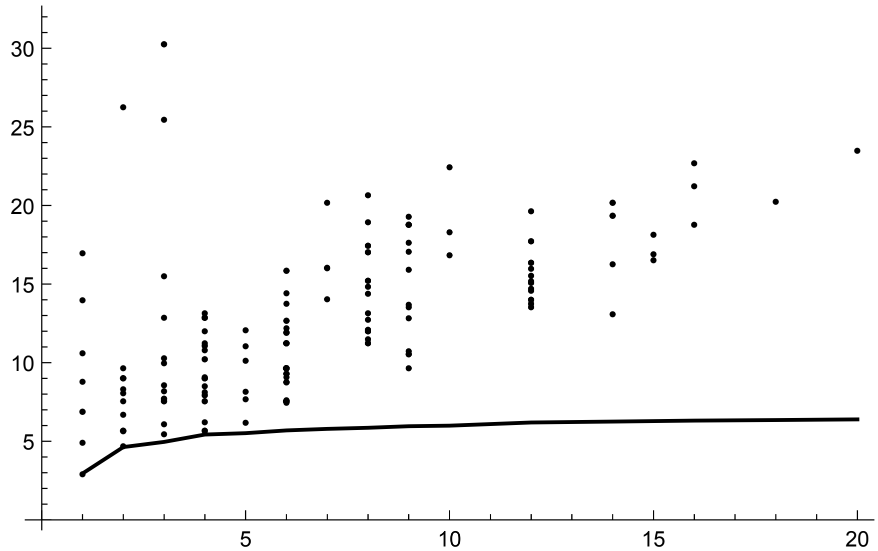

FIGURE 8.1. Points $(\chi_1(e), \mathfrak{d}(G,\chi))$ present lower bounds from Tables 8.1–8.8. The piecewise-linear curve plots lower bounds from [PM11]. Both the points and the curve assume the Artin conjecture and Riemann hypothesis for the relevant L-functions.

blocks of rows are indexed by increasing j. There is no block to print if mTj has no faithful irreducible characters. For example, there is no block to print for groups having noncyclic center, such as $4T2 = V = C_2 \times C_2$ or $8T9 = D_4 \times C_2$. Also the block belonging to mTj is omitted if the abstract group G underlying mTj has appeared earlier. For example $G = S_4$ has four transitive realization in degrees $m \leq 8$, namely $4T5$, $6T7$, $6T8$, and $8T14$; there is correspondingly a $4T5$ line on our tables, but no $6T7$, $6T8$, or $8T14$ lines.

8.2.1. *Top row of the G-block.* The top row in the G-block is different from the other rows, as it gives information corresponding to the abstract group G. Instead of referring to a faithful irreducible character, as the other lines do, many of its entries are the corresponding quantities for the regular character ϕ_G. The first four entries are a common name for the group G (if there is one), the order $\phi_G(e) = |G|$, the symbol TW if G is known to have the universal tame-wild property as defined in [JR14b], and finally k, N. Here, k is the size of the rational character table, and N is number of vertices of the polytope P_G discussed in §6.2, or a dash if we did not compute N. The last four entries are the smallest root discriminant of a Galois G field, the factored form of the corresponding discriminant, a cutoff B for which the set $\mathcal{K}(G;B)$ is known, and the size $|\mathcal{K}(G;B)|$.

8.2.2. *Remaining rows of the G-block.* Each remaining line of the G-block corresponds to a type (G,χ). However the number of rows in the G-block is typically substantially less than the number of faithful irreducible characters of G, as we list only one representative of each $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \times \mathrm{Out}(G)$ orbit of such characters. As an example, S_6 has eleven characters, all rational. Of the nine which are faithful, there are three which are fixed by the nontrivial element of $\mathrm{Out}(S_6)$ and the others

TABLE 8.1. Artin L-functions with small conductor from groups in degrees 2, 3, 4, and 5

G	n_1	z	$[-\check{\chi}, \hat{\chi}]$	$\mathfrak{d}$	δ_1	Δ_1	pos'n	β	B^β	#
C_2	2	TW	2, 2		1.73	3^*			100	6086
2T1	1		$[-1,-1]$	2.97_ℓ	3.00	3	1	2.00	10000.00	6086
C_3	3	TW	2, 2		3.66	7^*			500	1772
3T1	1	$\sqrt{-3}$	$[-1,-1]$	6.93_ℓ	7.00	7	1	1.50	11180.34	1772
S_3	6	TW	3, 4		4.80	23^*			250	24484
3T2	2		$[-1,0]$	4.74_ℓ	4.80	23	1	1.00	250.00	13329
C_4	4	TW	3, 4		3.34	5^*			150	2668
4T1	1	i	$[-2,0]$	4.96_S	5.00	5	1	$1.33_\bullet$	796.99	489
D_4	8	TW	5, 10		6.03	3^*7^*			150	31742
4T3	2		$[-2,0]$	5.74_q	6.24	$3{\cdot}13$	2	1.00	150.00	9868
A_4	12	TW	3, 4		10.35	2^*7^*			150	846
4T4	3		$[-1,0]$	7.60_q	14.64	$2^6 7^2$	1	1.00	150.00	270
S_4	24	TW	5, 12		13.56	2^*11^*			150	14587
6T8	3		$[-1,1]$	8.62_G	11.30	$2^2 19^2$	4	$0.89_\bullet$	85.96	779
4T5	3		$[-1,1]$	5.49_p	6.12	229	9	0.67	28.23	1603
C_5	5	TW	2, 2		6.81	11^*			200	49
5T1	1	ζ_5	$[-1,-1]$	10.67_ℓ	11.00	11	1	1.25	752.12	49
D_5	10	TW	3, 4		6.86	47^*			200	3622
5T2	2	$\sqrt{5}$	$[-1,0]$	6.73_q	6.86	47	1	1.00	200.00	3219
F_5	20	TW	4, 8		11.08	2^*5^*			200	3010
5T3	4		$[-1,0]$	10.28_q	13.69	$2^4 13^3$	2	1.00	200.00	2066
A_5	60	TW	4, 8		18.70	2^*17^*			85	473
5T4	4		$[-1,1]$	8.18_g	11.66	$2^6 17^2$	1	0.75	27.99	46
6T12	5		$[-1,1]$	10.18_p	12.35	$2^6 67^2$	3	0.80	34.96	216
12T33	3	$\sqrt{5}$	$[-2,1]$	10.34_g	26.45	$2^6 17^2$	1	0.83	40.54	18
S_5	120	TW	7, 40		24.18	$2^*3^*5^*$			85	2080
5T5	4		$[-1,2]$	6.28_ℓ	6.33	1609	103	0.50	9.22	45
10T12	4		$[-2,1]$	10.28_V	18.72	$5^2 17^3$	21	0.75	27.99	15
10T13	5		$[-1,1]$	12.13_V	16.27	$2^4 3^2 89^2$	6	0.80	34.96	592
6T14	5		$[-1,1]$	11.09_g	17.78	$2^6 3^4 7^3$	14	0.80	34.96	186
20T35	6		$[-2,1]$	12.26_g	18.18	$2^4 3^3 17^4$	12	0.83	40.54	110

form three two-element orbits. Thus the S_6-block has six rows. In general, the information on a (G, χ) row comes in three parts, which we now describe in turn.

First four columns. The first column gives the lexicographically first permutation group mTj for which the corresponding permutation character has χ as a rational constituent. Then $n_1 = \chi_1(e)$ is the degree of an absolutely irreducible character χ_1 such that χ is the sum of its conjugates. The number n_1 is superscripted by the size of the $\mathrm{Out}(G)$ orbit of χ, in the unusual case when this orbit size is not 1. Next, the complex number z is a generator for the field generated by the values of the character χ_1, with no number printed in the common case that χ_1 is rational-valued. The last entry gives the interval $[-\check{\chi}, \hat{\chi}]$, where $\check{\chi}$ and $\hat{\chi}$ are the numbers introduced in the beginning of Section 5. In the range presented, the data of mTj, n_1, z, and $[-\check{\chi}, \hat{\chi}]$ suffice to distinguish Galois types (G, χ) from each other.

TABLE 8.2. Artin L-functions of small conductor from sextic groups

G	n_1	z	$[-\check{\chi}, \hat{\chi}]$	$\mathfrak{d}$	δ_1	Δ_1	pos'n	β	B^β	#
C_6	6	TW	4, 6		5.06	7^*			200	9609
6T1	1	$\sqrt{-3}$	$[-2, 1]$	6.93_P	7.00	7	1	$1.20_\bullet$	577.08	617
D_6	12	TW	6, 14		8.06	3^*5^*			150	46197
6T3	2		$[-2, 1]$	7.60_G	9.33	$3{\cdot}29$	6	$1.00_\bullet$	150.00	10242
S_3C_3	18		6, 17		10.06	$2^*3^*7^*$			200	9420
6T5	2	$\sqrt{-3}$	$[-2, 1]$	5.69_{q*}	7.21	$2^2 13$	4	0.75	53.18	503
A_4C_2	24		6, 16		12.31	2^*7^*			150	6676
6T6	3		$[-3, 1]$	7.60_p	8.60	$7^2 13$	3	0.67	28.23	98
S_3^2	36		9, 69		15.53	2^*19^*			200	45117
6T9	4		$[-2, 1]$	7.98_{q*}	14.83	$2^4 5^2 11^2$	27	0.75	53.18	824
$C_3^2 \rtimes C_4$	36	TW	5, 16		23.57	3^*5^*			150	331
6T10	4^2		$[-2, 1]$	7.98_{q*}	17.80	$2^{11} 7^2$	2	0.75	42.86	33
S_4C_2	48		10, 96		16.13	2^*23^*			150	70926
6T11	3^2		$[-3, 1]$	6.14_g	6.92	$2^2 83$	7	0.67	28.23	3694
$C_3^2 \rtimes D_4$	72	TW	9, 105		21.76	3^*11^*			150	8536
6T13	4^2		$[-2, 2]$	7.60_p	7.90	$3^2 433$	52	0.50	12.25	41
12T36	4^2		$[-2, 1]$	11.29_P	23.36	$3^5 5^2 7^2$	18	0.75	42.86	106
A_6	360	TW	6, 28		31.66	2^*3^*			60	26
6T15	5^2		$[-1, 2]$	7.71_ℓ	12.35	$2^6 67^2$	8	0.60	11.67	0
10T26	9		$[-1, 1]$	17.69_g	28.20	$2^{18} 3^{16}$	1	0.89	38.07	7
30T88	10		$[-2, 1]$	18.34_g	30.61	$2^{24} 3^{16}$	1	0.90	39.84	4
36T555	8	$\sqrt{5}$	$[-2, 1]$	20.70_g	42.81	$2^{18} 3^{16}$	1	0.94	46.45	3
S_6	720		11, 596		33.50	$2^*3^*5^*$			60	99
12T183	5^2		$[-3, 2]$	8.21_v	11.53	$11^2 41^2$	6	0.60	11.67	1
6T16	5^2		$[-1, 3]$	6.23_ℓ	6.82	14731	53	0.40	5.14	0
10T32	9		$[-1, 3]$	10.77_v	*16.60*	$2^{15} 11^3 13^3$	74	0.67	15.33	0
20T145	9		$[-3, 1]$	19.33_g	31.25	$2^6 5^6 73^4$	16	0.89	38.07	4
30T176	10^2		$[-2, 2]$	16.88_v	24.22	$11^4 41^6$	6	0.80	26.46	1
36T1252	16		$[-2, 1]$	22.73_g	35.46	$2^{36} 3^8 7^{12}$	5	0.94	46.45	11

Middle four columns. The next four columns focus on minimal root conductors. In the first entry, $\mathfrak{d}$ is the best conditional lower bound we obtained for root conductors, and the subscript $i \in \{\ell, s, q, g, p, v\}$ gives information on the corresponding auxiliary character ϕ. The first four possibilities refer to the methods of Section 5, namely *linear*, *square*, *quadratic*, and *Galois*. The last two, p and v, indicate a *permutation* character and a character coming from a *vertex* of the polytope P_G. The best ϕ of the ones we inspect is always at a vertex, except in the three cases on Table 8.2 where $*$ is appended to the subscript. Capital letters S, Q, G, P, and V also appear as subscripts. These occur only for groups marked with TW, and indicate that the tame-wild principle improved the lower bound. For most groups with fifteen or more classes, it was prohibitive to calculate all vertices, and the best of the other methods is indicated.

When the second entry is in roman type, it is the minimal root conductor and the third entry is the minimal conductor in factored form. When the second entry is in italic type, then it is the smallest currently known root conductor. The fourth

TABLE 8.3. Artin L-functions of small conductor from septic groups

G	n_1	z	$[-\check{\chi}, \widehat{\chi}]$	$\mathfrak{d}$	δ_1	Δ_1	pos'n	β	B^β	#
C_7	7	TW	2, 2		17.93	29^*			200	15
7T1	1	ζ_7	$[-1,-1]$	14.03_ℓ	29.00	29	1	1.17	483.65	15
D_7	14	TW	3, 4		8.43	71^*			200	2078
7T2	2	ζ_7^+	$[-1,0]$	8.38_q	8.43	71	1	1.00	200.00	1948
$C_7{\rtimes}C_3$	21	TW	3, 4		31.64	2^*73^*			100	11
7T3	3	$\sqrt{-7}$	$[-1,0]$	25.50_q	34.93	$2^3 73^2$	1	1.00	100.00	8
F_7	42	TW	5, 12		15.99	2^*7^*			75	342
7T4	6		$[-1,0]$	14.47_q	18.34	$11^3 13^4$	2	1.00	75.00	287
$GL_3(2)$	168	TW	5, 14		32.25	$2^*3^*11^*$			45	19
42T37	3	$\sqrt{-7}$	$[-2,2]$	15.55_G	26.06	$7^2 19^2$	7	$0.89_\bullet$	29.48	1
7T5	6		$[-1,2]$	9.36_p	11.23	$13^2 109^2$	4	0.67	12.65	1
8T37	7		$[-1,1]$	14.10_g	32.44	$3^8 7^8$	11	0.86	26.12	0
21T14	8		$[-1,1]$	14.90_g	23.16	$2^6 3^6 11^6$	1	0.88	27.96	1
A_7	2520		8, 115		39.52	$2^*3^*7^*$			45	1
7T6	6		$[-1,3]$	9.13_ℓ	12.54	$3^6 73^2$	26	0.50	6.71	0
15T47	14		$[-1,2]$	19.39_g	*36.05*	$3^{24} 5^6$	4	0.86	26.12	0
21T33	14		$[-1,2]$	19.39_g	*31.07*	$3^{18} 17^{10}$	2	0.86	26.12	0
42T294	15		$[-1,3]$	18.18_v	*35.73*	$2^{12} 3^{20} 7^{12}$	1	0.80	21.02	0
70	10	$\sqrt{-7}$	$[-4,2]$	22.49_g	*41.21*	$2^9 3^{14} 7^8$	1	0.90	30.75	0
42T299	21		$[-3,1]$	26.95_g	*38.33*	$2^{18} 3^{30} 7^{16}$	1	0.95	37.54	0
70	35		$[-1,1]$	*28.79*$_g$	*41.28*	$2^{30} 3^{50} 7^{28}$	1	$0.97_\circ$	40.36	0
S_7	5040		15, –		40.49	$2^*3^*5^*$			35	0
7T7	6		$[-1,4]$	7.50_ℓ	7.55	184607		0.33	3.27	0
14T46	6		$[-4,3]$	*7.66*$_p$	*17.02*	$2^2 7^5 19^2$	194	0.50	5.92	0
30T565	14		$[-2,4]$	16.32_p	*26.02*	$2^{20} 5^3 8$	2	0.71	12.67	0
30T565	14		$[-4,2]$	20.24_g	*30.98*	$2^{14} 71^9$	46	0.86	21.06	0
42T413	14		$[-6,2]$	20.24_g	*38.27*	$2^{20} 3^{12} 11^{10}$	6	0.86	21.06	0
21T38	14		$[-1,6]$	13.12_p	*22.02*	$2^{24} 3^{12} 29^4$	170	0.57	7.63	0
42T412	15		$[-3,5]$	16.96_p	*32.90*	$3^{12} 5^5 11^{13}$	24	0.67	10.70	0
42T411	15		$[-5,3]$	16.56_g	*29.92*	$2^{30} 3^{12} 17^6$	3	0.80	17.19	0
70	20		$[-4,2]$	23.53_g	*35.18*	$2^{34} 53^{12}$	2	0.90	24.53	0
42T418	21		$[-3,3]$	20.24_g	*33.42*	$2^{41} 3^{18} 17^9$	3	0.86	21.06	0
84	21		$[-3,1]$	28.27_g	*39.59*	$2^{38} 3^{18} 7^{16}$	4	0.95	29.55	0
70	35		$[-1,5]$	*25.92*$_p$	*40.71*	$2^{61} 3^{30} 7^{28}$	4	0.86	21.06	0
126	35		$[-5,1]$	*30.23*$_g$	*43.26*	$2^{54} 3^{42} 5^{30}$		$0.97_\circ$	31.62	0

entry gives the position of the source number field on the complete list ordered by Galois root discriminant. This information lets readers obtain further information from [JR14a], such as a defining polynomial and details on ramification.

Last three columns. The quantity β is the exponent we are using to pass from Galois number fields to Artin representations. Writing $\widehat{\underline{\alpha}} = \widehat{\underline{\alpha}}(G, \chi, \phi_G)$ and $\underline{\alpha} = \underline{\alpha}(G, \chi, \phi_G)$, one has the universal relation $\widehat{\underline{\alpha}} \leq \underline{\alpha}$. When equality holds then the common number is printed. To indicate that inequality holds, an extra symbol is printed. When we know that G satisfies TW then we can use larger exponent and $\underline{\alpha}_\bullet$ is printed. Otherwise we use the smaller exponent and $\widehat{\underline{\alpha}}_\circ$ is printed. The column

TABLE 8.4. Artin L-functions of small conductor from octic groups

G	n_1	z	$[-\check{\chi}, \hat{\chi}]$	$\mathfrak{d}$	δ_1	Δ_1	pos'n	β	B^β	#
C_8	8	TW	4, 8		11.93	17^*			125	198
8T1	1	ζ_8	$[-4, 0]$	8.84_S	17.00	17	1	$1.14_\bullet$	249.15	41
Q_8	8	TW	5, 10		18.24	2^*3^*			100	72
8T5	2		$[-2, 0]$	26.29_S	48.00	$2^8 3^2$	2	$1.33_\bullet$	464.16	41
D_8	16	TW	6, 20		9.75	5^*19^*			125	6049
8T6	2	$\sqrt{2}$	$[-4, 0]$	9.07_q	9.75	$5 \cdot 19$	1	1.00	125.00	2296
$C_8 \rtimes C_2$	16		7, 24		9.32	3^*5^*			125	672
8T7	2	i	$[-4, 0]$	9.07_q	15.00	$3^2 5^2$	1	1.00	125.00	75
QD_{16}	16		6, 20		10.46	2^*3^*			125	1664
8T8	2	$\sqrt{-2}$	$[-4, 0]$	9.07_q	16.97	$2^5 3^2$	1	1.00	125.00	155
$Q_8 \rtimes C_2$	16		9, 32		9.80	2^*3^*			100	3366
8T11	2	i	$[-4, 0]$	9.07_q	10.95	$2^3 3 \cdot 5$	3	1.00	100.00	825
$\mathrm{SL}_2(3)$	24	TW	5, 14		29.84	163^*			250	681
24T7	2		$[-2, 1]$	65.51_P	163.00	163^2	1	$1.20_\bullet$	754.27	94
8T12	2	$\sqrt{-3}$	$[-4, 1]$	8.09_p	12.77	163	1	0.75	62.87	78
	32		11, 74		13.79	2^*5^*			125	11886
8T15	4		$[-4, 0]$	12.92_q	16.12	$2^4 5^2 13^2$	4	1.00	125.00	3464
	32		9, 58		13.56	5^*11^*			125	766
8T16	4		$[-4, 0]$	12.92_q	16.58	$5^4 11^2$	1	1.00	125.00	129
$C_4 \wr C_2$	32		10, 90		13.37	2^*5^*			125	2748
8T17	2^2	i	$[-4, 2]$	$\mathit{5.74}_p$	8.25	$2^2 17$	6	$0.50_\circ$	11.18	3
	32		9, 58		14.05	2^*			125	2720
8T19	4		$[-4, 0]$	12.92_q	19.03	2^{17}	1	1.00	125.00	1282
	32		17, 806		18.42	$2^*3^*5^*$			100	3284
8T22	4		$[-4, 0]$	12.92_q	20.49	$2^4 3^2 5^2 7^2$	3	1.00	100.00	1162
$\mathrm{GL}_2(3)$	48		7, 41		16.52	2^*43^*			100	2437
24T22	2	$\sqrt{-2}$	$[-4, 2]$	$\mathit{5.74}_v$	*16.82*	283	2	$0.50_\circ$	10.00	0
8T23	4		$[-4, 1]$	9.07_p	9.95	$3^4 11^2$	4	0.75	31.62	99
$C_2^3 \rtimes C_7$	56	TW	3, 4		17.93	29^*			200	28
8T25	7		$[-1, 0]$	16.10_q	17.93	29^6	1	1.00	200.00	27
	64		16, –		20.37	2^*5^*			125	10317
8T26	4^2		$[-4, 2]$	9.07_p	*12.85*	$3^2 5^2 11^2$	7	$0.50_\circ$	11.18	0
$C_2 \wr C_4$	64		11, 206		19.44	2^*			125	2482
8T27	4^2		$[-4, 2]$	9.07_p	10.60	$5^3 101$	19	0.50	11.18	1
$C_2 \wr C_2^2$	64		16, –		19.41	2^*7^*			125	11685
8T29	4^2		$[-4, 2]$	9.07_p	10.13	$2^4 3^2 73$	28	0.50	11.18	1

B^β gives the corresponding upper bound on our complete list of root conductors. Finally the column # gives $|\mathcal{L}(G, \chi; B^\beta)|$, the length of the complete list of Artin L-functions we have identified. For the L-functions themselves, we refer to [LMF].

Table 8.5. Artin L-functions of small conductor from octic groups

G	n_1	z	$[-\check{\chi}, \hat{\chi}]$	$\mathfrak{d}$	δ_1	Δ_1	pos'n	β	B^β	#
	64		11, 206		19.44	2^*			125	1217
8T30	4^2		$[-4,2]$	9.07_p	$\mathit{14.57}$	$5^3 19^2$	3	$0.50_\circ$	11.18	0
	96		9, 49		34.97	$2^*5^*13^*$			250	5520
8T32	4		$[-4,1]$	9.12_g	22.80	$2^6 5^2 13^2$	2	0.75	62.87	180
24T97	4	$\sqrt{-3}$	$[-8,1]$	$\mathit{13.19}_g$	43.30	$2^6 5^2 13^3$	2	$0.88_\circ$	125.37	112
	96		8, 44		30.01	$2^*5^*7^*$			150	791
8T33	6^2		$[-2,2]$	11.29_p	25.14	$5^3 7^4 29^2$	12	0.67	28.23	3
	96		10, 92		27.28	$2^*3^*31^*$			110	1915
8T34	6		$[-2,2]$	11.29_p	22.61	$31^3 67^2$	64	0.67	22.96	1
$C_2 \wr D_4$	128		20, –		22.91	$2^*3^*13^*$			125	14369
8T35	4^4		$[-4,2]$	9.07_p	9.45	$5^2 11{\cdot}29$	110	0.50	11.18	9
$C_2^3 \rtimes F_{21}$	168		5, 14		31.64	2^*73^*			200	342
8T36	7		$[-1,1]$	16.06_p	21.03	$2^6 73^4$	1	0.86	93.82	120
24T283	7	$\sqrt{-3}$	$[-2,1]$	$\mathit{20.23}_p$	38.55	$2^6 7^{11}$	2	$0.93_\circ$	136.98	81
$C_2 \wr A_4$	192		12, 700		37.27	$2^*5^*7^*$			250	13649
8T38	4^2		$[-4,2]$	8.56_v	15.20	$2^6 7^2 17$	11	0.50	15.81	1
24T288	4^2	$\sqrt{-3}$	$[-8,4]$	$\mathit{10.84}_p$	$\mathit{23.95}$	$2^6 3{\cdot}5{\cdot}7^3$	66	0.50	15.81	0
	192		13, 559		32.35	2^*23^*			100	1193
8T39	4^2		$[-4,2]$	5.74_p	8.71	$2^4 359$	49	0.50	10.00	8
24T333	8		$[-8,1]$	$\mathit{15.28}_g$	39.94	$2^{10} 43^6$	2	$0.88_\circ$	56.23	16
	192		13, 559		29.71	2^*23^*			100	2001
8T40	4^2		$[-4,2]$	$\mathit{5.74}_p$	$\mathit{13.04}$	$2^2 5^2 17^2$	9	$0.50_\circ$	10.00	0
24T332	8		$[-8,1]$	$\mathit{15.28}_g$	29.71	$2^{12} 23^6$	1	$0.88_\circ$	56.23	47
	192		14, 1210		28.11	2^*11^*			100	4723
12T108	6^2		$[-2,2]$	$\mathit{11.29}_p$	20.78	$2^{12} 3^9$	5	0.67	21.54	2
8T41	6^2		$[-2,2]$	11.29_p	13.01	$5^3 197^2$	13	0.67	21.54	40
$A_4 \wr C_2$	288		10, 178		32.18	2^*37^*			135	1362
8T42	6		$[-2,3]$	11.29_p	11.58	$5^3 139^2$	76	0.50	11.62	1
18T112	9		$[-3,1]$	$\mathit{17.10}_g$	35.11	$2^{24} 13^6$	2	0.89	78.28	66
12T128	9		$[-3,3]$	13.59_p	22.52	$7^6 233^3$	56	0.67	26.32	6
24T703	6	$\sqrt{-3}$	$[-4,4]$	$\mathit{13.79}_p$	$\mathit{32.18}$	$2^4 37^5$	1	0.67	26.32	0
$C_2 \wr S_4$	384		20, –		31.38	5^*197^*			100	6400
8T44	4^4		$[-4,2]$	5.74_p	7.53	$5{\cdot}643$	391	0.50	10.00	26
24T708	8^2		$[-8,4]$	$\mathit{13.19}_p$	$\mathit{25.55}$	$2^{12} 5^2 11^6$	4	0.50	10.00	0

9. Discussion of tables

In this section, we discuss four topics, each of which makes specific reference to parts of the tables of the previous section. Each of the topics also serves the general purpose of making the tables more readily understandable.

9.1. **Comparison of first Galois root discriminants and root conductors.**

Suppose first, for notational simplicity, that G is a group for which all irreducible complex characters take rational values only. When one fixes K^{gal} with $\mathrm{Gal}(K^{\mathrm{gal}}/\mathbf{Q}) \cong G$ and lets χ runs over all the irreducible characters of G, the root

Table 8.6. Artin L-functions of small conductor from octic groups

G	n_1	z	$[-\check{\chi}, \hat{\chi}]$	$\mathfrak{d}$	δ_1	Δ_1	pos'n	β	B^β	#
	576		16, –		29.35	2^*3^*			100	2664
12T161	6		$[-2,3]$	$\mathit{7.60}_p$	$\mathit{19.04}$	$2^8 3^3 83^2$	1179	0.50	10.00	0
8T45	6		$[-2,3]$	7.60_p	15.48	$2^{14}29^2$	110	0.50	10.00	0
18T179	9		$[-3,1]$	18.84_g	29.69	$2^{12}5^6 23^4$	16	0.89	59.95	121
12T165	9		$[-3,3]$	10.60_p	17.20	$2^6 19^3 67^3$	161	0.67	21.54	12
18T185	9^2		$[-3,3]$	$\mathit{13.74}_p$	$\mathit{22.69}$	$2^{12}3^{11}13^3$	6	0.67	21.54	0
24T1504	12		$[-4,4]$	$\mathit{16.40}_p$	$\mathit{35.03}$	$2^8 5^6 31^8$	51	0.67	21.54	0
	576		11, 522		49.75	$3^*5^*7^*$			100	153
8T46	6		$[-2,3]$	9.66_p	19.51	$3^6 5^4 11^2$	42	0.50	10.00	0
12T160	6		$[-2,3]$	$\mathit{7.60}_p$	$\mathit{27.27}$	$2^{23}7^2$	11	0.50	10.00	0
16T1030	9		$[-3,1]$	18.84_g	35.40	$2^{22}3^6 13^4$	6	0.89	59.95	50
18T184	9		$[-3,1]$	18.84_g	35.50	$2^{12}5^7 23^4$	7	0.89	59.95	32
24T1506	12		$[-4,4]$	$\mathit{16.40}_p$	$\mathit{55.16}$	$2^{20}3^{18}5^9$	4	0.67	21.54	0
36T766	9	i	$[-6,2]$	$\mathit{18.84}_g$	58.55	$2^{36}7^6$	3	0.89	59.95	2
$S_4 \wr C_2$	1152		20, –		35.05	$2^*5^*41^*$			150	23694
12T200	6		$[-2,4]$	$\mathit{7.60}_p$	$\mathit{15.62}$	$5^3 11^2 31^2$	20668	$0.33_\circ$	5.31	0
8T47	6		$[-2,4]$	7.60_p	10.51	$2^9 2633$	20566	0.33	5.31	0
12T201	6		$[-4,3]$	$\mathit{7.60}_p$	$\mathit{19.19}$	$3^7 151^2$	21	0.50	12.25	0
12T202	6		$[-4,3]$	$\mathit{7.60}_p$	$\mathit{16.51}$	$3^{10}7^3$	12	0.50	12.25	0
18T272	9		$[-3,3]$	$\mathit{9.70}_p$	19.73	$3^6 853^3$	450	0.67	28.23	44
18T274	9		$[-3,3]$	15.95_p	27.07	$2^{12}5^7 29^3$	105	0.67	28.23	3
18T273	9		$[-3,3]$	$\mathit{12.88}_p$	$\mathit{30.86}$	$2^{16}3^{18}$	5	$0.67_\circ$	28.23	0
16T1294	9		$[-3,3]$	10.60_p	13.16	$43^3 53^3$	39	0.67	28.23	295
36T1946	12		$[-4,4]$	$\mathit{14.77}_p$	$\mathit{32.80}$	$2^{10}13^7 17^6$	16	0.67	28.23	0
24T2821	12		$[-4,4]$	$\mathit{16.03}_p$	26.48	$2^{16}5^6 41^5$	1	0.67	28.23	1
36T1758	18		$[-6,2]$	$\mathit{20.29}_g$	36.08	$2^{24}5^9 41^9$	1	0.89	85.96	1222

discriminant δ_{Gal} is just the weighted multiplicative average $\prod_\chi = \delta_\chi^{\chi(e)^2/|G|}$. Deviation of a root conductor δ_χ from δ_{Gal} is caused by nonzero values of χ. When $\chi(e)$ is large and $[-\check{\chi}, \hat{\chi}]$ is small, δ_χ is necessarily close to δ_{Gal}. One can therefore think generally of δ_{Gal} as a first approximation to δ_χ. The general principle of δ_{Gal} approximating δ_χ applies to groups G with irrational characters as well.

Our first example of S_5 illustrates both how the principle $\delta_{\mathrm{Gal}} \approx \delta_\chi$ is reflected in the tables, and how it tends to be somewhat off in the direction that $\delta_{\mathrm{Gal}} > \delta_\chi$. For a given K^{gal}, the variance of its δ_χ about its δ_{Gal} is substantial and depends on the details of the ramification in K^{gal}. There are many K^{gal} with root discriminant near the minimal root discriminant, all of which are possible sources of minimal root conductors. It is therefore expected that the minimal conductors $\delta_1(S_5, \chi) = \min \delta_\chi$ printed in the table, 6.33, 18.72, 16.27, 17.78, and 18.18, are substantially less than the printed minimal root discriminant $\delta_1(S_5, \phi_{120}) \approx 24.18$. As groups G get larger, one can generally expect tighter clustering of the $\delta_1(G, \chi)$ about $\delta_1(G, \phi_G)$. One can see the beginning of this trend in our partial results for S_6 and S_7.

9.2. **Known and unknown minimal root conductors.** Our method of starting with a complete list of Galois fields is motivated by the principle from the previous

TABLE 8.7. Artin L-functions of small conductor from nonic groups

G	n_1	z	$[-\check{\chi},\hat{\chi}]$	$\mathfrak{d}$	δ_1	Δ_1	pos'n	β	B^β	#
C_9	9	TW	3, 4		13.70	19^*			200	48
9T1	1	ζ_9	$[-3,0]$	17.02_Q	19.00	19	1	$1.13_\bullet$	387.85	26
D_9	18	TW	4, 8		12.19	2^*59^*			200	699
9T3	2	ζ_9^+	$[-3,0]$	9.70_q	14.11	199	3	1.00	200.00	638
3_-^{1+2}	27		6, 12		31.18	7^*13^*			200	32
9T6	3	$\sqrt{-3}$	$[-3,0]$	30.32_q	38.70	$7^3 13^2$	1	1.00	200.00	14
3_+^{1+2}	27	TW	6, 12		50.20	3^*19^*			200	16
9T7	3	$\sqrt{-3}$	$[-3,0]$	30.32_q	64.08	$3^6 19^2$	1	1.00	200.00	12
$3^{1+2}.2$	54		7, 34		17.01	2^*3^*			200	981
9T10	6		$[-3,0]$	15.90_q	17.49	31^5	2	1.00	200.00	741
	54		7, 34		16.83	3^*7^*			200	880
9T11	6		$[-3,0]$	15.90_q	19.01	$3^9 7^4$	1	1.00	200.00	805
	54		8, 42		16.72	$2^*3^*5^*$			200	2637
9T12	3	$\sqrt{-3}$	$[-3,2]$	7.75_p	10.71	$2^2 307$	7	0.67	34.20	256
18T24	3	$\sqrt{-3}$	$[-3,1]$	*10.03_g*	20.08	$2^2 3^4 5^2$	1	$0.83_\circ$	82.70	77
M_9	72		6, 20		29.72	2^*3^*			100	27
9T14	8		$[-1,0]$	17.50_q	31.59	$2^{24} 3^{10}$	1	1.00	100.00	26
$C_3^2{\rtimes}C_8$	72	TW	5, 16		25.41	2^*3^*			100	19
9T15	8		$[-1,0]$	17.50_q	25.41	$2^{31} 3^4$	1	1.00	100.00	16
$C_3 \wr C_3$	81		9, 59		75.41	3^*19^*			500	131
9T17	3^3	$\sqrt{-3}$	$[-3,3]$	12.92_q	30.14	$7^2 13{\cdot}43$	22	0.50	22.36	0
$C_3^2{\rtimes}D_6$	108		11, 262		22.06	3^*23^*			150	12002
18T55	6		$[-3,1]$	*11.98_g*	24.38	$2^8 3^8 5^3$	4	$0.83_\circ$	65.07	229
9T18	6		$[-3,2]$	9.70_p	13.46	$3^3 7^2 67^2$	53	0.67	28.23	216
$C_3^2{\rtimes}QD_{16}$	144		8, 62		23.41	3^*7^*			100	488
9T19	8		$[-1,2]$	*12.13_v*	25.65	$3^{13} 7^6$	1	0.75	31.62	14
18T68	8		$[-2,1]$	*14.44_g*	25.65	$3^{13} 7^6$	1	$0.88_\circ$	56.23	48
$C_3 \wr S_3$	162		13, 2004		29.89	3^*			200	1617
9T20	3^3	$\sqrt{-3}$	$[-3,3]$	7.75_p	11.17	$7{\cdot}199$	23	0.50	14.14	20
18T86	3^3	$\sqrt{-3}$	$[-3,3]$	*8.23_v*	*19.48*	$2^2 43^2$	5	0.50	14.14	0
	162		11, 223		24.90	$2^*3^*5^*$			100	597
9T21	6^3		$[-3,3]$	9.70_p	15.58	$5^2 83^3$	2	0.50	14.14	0
	162		10, 205		26.46	3^*			100	180
9T22	6^3		$[-3,3]$	9.70_p	17.21	$2^6 7^4 13^2$	6	0.50	10.00	0
$C_3^2{\rtimes}\mathrm{SL}_2(3)$	216		7, 44		49.57	349^*			100	37
9T23	8		$[-1,2]$	11.29_p	23.39	547^4	10	0.75	31.62	3
24T569	8	$\sqrt{-3}$	$[-2,1]$	18.99_g	38.84	349^5	1	0.94	74.99	20
	324		17, –		30.64	$2^*3^*11^*$			100	1816
18T129	6^3		$[-3,3]$	*9.34_p*	*22.88*	$2^9 23^4$	16	0.50	14.14	0
9T24	6^3		$[-3,3]$	9.70_p	15.84	$3^7 5^2 17^2$	399	0.50	14.14	0

subsection that the Galois root discriminant δ_{Gal} is a natural first approximation to δ_χ. Indeed, as the tables show via nonzero entries in the # column, this general method suffices to obtain a non-empty initial segment for most (G,χ). As our focus

TABLE 8.8. Artin L-functions of small conductor from nonic groups

G	n_1	z	$[-\check{\chi},\hat{\chi}]$	$\mathfrak{d}$	δ_1	Δ_1	pos'n	β	B^β	#
	324		9, 116		29.96	2^*3^*			100	107
9T25	6		$[-3,3]$	12.73_p	22.25	$2^6 3^8 17^2$	59	0.50	10.00	0
18T141	6		$[-3,3]$	$\mathit{9.34}_p$	$\mathit{30.81}$	$3^8 19^4$	9	0.50	10.00	0
12T133	4	$\sqrt{-3}$	$[-4,2]$	11.15_g	19.34	$2^6 3^7$	1	0.75	31.62	10
12T132	4^2	$\sqrt{-3}$	$[-4,2]$	$\mathit{11.15}_g$	$\mathit{33.50}$	$2^6 3^9$	1	0.75	31.62	0
18T142	12		$[-3,3]$	19.68_p	30.57	$2^{18} 3^{26}$	3	0.75	31.62	2
$C_3^2 \rtimes \mathrm{GL}_2(3)$	432		10, 206		27.88	3^*11^*			76	453
9T26	8		$[-1,2]$	11.54_g	17.59	$2^6 523^3$	14	0.75	25.74	17
18T157	8		$[-2,2]$	$\mathit{12.14}_p$	19.04	$3^7 53^4$	16	0.75	25.74	3
24T1334	16		$[-2,1]$	21.27_g	26.68	$3^{26} 11^{10}$	1	0.94	57.98	134
$S_3 \wr C_3$	648		14, 3706		33.56	$2^*5^*13^*$			150	1677
18T197	6^2		$[-3,3]$	$\mathit{9.70}_v$	$\mathit{19.71}$	$7^4 29^3$	1	0.50	12.25	0
18T202	6		$[-4,3]$	$\mathit{9.70}_v$	$\mathit{27.73}$	$2^6 3^9 19^2$	49	0.50	12.25	0
9T28	6		$[-3,4]$	9.70_p	12.20	$3^8 503$	335	0.33	5.31	0
12T176	8		$[-4,2]$	12.05_g	21.75	$11^4 43^4$	4	0.75	42.86	57
36T1102	12		$[-4,3]$	$\mathit{15.58}_v$	29.90	$2^6 5^{10} 13^8$	1	0.75	42.86	2
18T206	12		$[-3,4]$	15.23_v	21.99	$2^6 7^{10} 29^4$	10	0.67	28.23	8
24T1539	8	$\sqrt{-3}$	$[-8,4]$	$\mathit{17.07}_v$	$\mathit{45.71}$	$2^{14} 3^{19}$	3	0.75	42.86	0
	648		13, 2206		40.81	$2^*3^*17^*$			200	838
9T29	6		$[-3,3]$	12.73_p	16.62	$2^8 7^2 41^2$	31	0.50	14.14	0
18T223	6		$[-3,3]$	$\mathit{9.70}_v$	$\mathit{30.14}$	$2^4 5^2 37^4$	71	0.50	14.14	0
24T1527	4	$\sqrt{-3}$	$[-4,2]$	12.05_g	32.34	$2^2 3^7 5^3$	18	0.75	53.18	43
12T175	4	$\sqrt{-3}$	$[-4,4]$	7.60_p	9.23	$11\cdot659$	164	0.50	14.14	14
36T1131	6	$\sqrt{-3}$	$[-6,6]$	$\mathit{11.95}_v$	$\mathit{36.04}$	$2^2 3^8 17^4$	6	0.50	14.14	0
36T1237	12		$[-3,3]$	$\mathit{17.78}_p$	44.72	$2^{22} 3^7 17^8$	1	0.75	53.18	6
18T219	12		$[-3,3]$	$\mathit{14.05}_v$	33.23	$3^{17} 107^5$	5	0.75	53.18	65
24T1540	8	$\sqrt{-3}$	$[-8,4]$	$\mathit{17.07}_v$	49.37	$2^{10} 3^{15} 7^4$	2	0.75	53.18	1
	648		13, 1322		30.37	2^*269^*			200	4001
9T30	6		$[-3,3]$	9.70_p	10.67	$11^2 23^3$	3	0.50	14.14	1
18T222	6		$[-3,3]$	$\mathit{9.70}_v$	$\mathit{23.27}$	$31^3 73^2$	57	0.50	14.14	0
12T178	8		$[-4,2]$	12.05_g	18.27	$2^{10} 59^4$	10	0.75	53.18	327
12T177	8^2		$[-4,2]$	$\mathit{12.05}_g$	24.98	$2^{10} 23^6$	22	0.75	53.18	173
36T1121	6	$\sqrt{3}$	$[-6,6]$	$\mathit{11.95}_v$	$\mathit{31.61}$	$2^8 7^2 43^3$	91	0.50	14.14	0
36T1123	12		$[-3,3]$	$\mathit{17.78}_p$	28.87	$2^{35} 5^{10}$	2	0.75	53.18	71
18T218	12		$[-3,3]$	14.05_v	18.33	$2^{10} 269^5$	1	0.75	53.18	453
$S_3 \wr S_3$	1296		22, –		36.26	2^*3^*			200	12152
18T320	6		$[-3,4]$	$\mathit{8.80}_p$	$\mathit{14.80}$	$5\cdot23^3 173$	8562	0.33	5.85	0
18T312	6		$[-4,3]$	$\mathit{8.79}_p$	$\mathit{17.45}$	$3^5 11^2 31^2$	343	0.50	14.14	0
9T31	6		$[-3,4]$	9.70_p	10.38	$31^2 1303$	10036	0.33	5.85	0
18T303	6		$[-4,3]$	$\mathit{8.79}_p$	$\mathit{18.34}$	$5^5 23^3$	4	0.50	14.14	0
12T213	8		$[-4,4]$	11.29_p	13.38	$5^2 7^4 131^2$	397	0.50	14.14	3
24T2895	8		$[-4,2]$	$\mathit{12.79}_g$	30.65	$2^8 3^4 5^6 7^4$	77	0.75	53.18	230
18T315	12		$[-3,4]$	13.59_p	22.81	$2^{10} 23^4 37^5$	72	0.67	34.20	105
36T2216	12		$[-3,4]$	$\mathit{13.79}_p$	29.13	$2^{10} 3^{17} 41^4$	78	0.67	34.20	11
36T2305	12		$[-4,3]$	$\mathit{15.14}_p$	31.73	$2^{18} 331^5$	58	0.75	53.18	151
36T2211	12		$[-6,6]$	$\mathit{14.64}_p$	$\mathit{37.29}$	$2^{16} 5^8 7^{10}$	55	0.50	14.14	0
36T2214	12		$[-4,3]$	$\mathit{15.14}_p$	32.07	$2^{22} 3^{24}$	11	0.75	53.18	136
24T2912	16		$[-8,4]$	$\mathit{18.82}_p$	35.12	$5^{12} 23^{12}$	4	0.75	53.18	40

is primarily on the first root conductor $\delta_1 = \delta_1(G, \chi)$, we do not pursue larger initial segments in these cases.

When the initial segment from our general method is empty, as reported by a 0 in the # column, we aim to nonetheless present the minimal root conductor δ_1. Suppose there are subgroups $H_m \subset H_k \subseteq G$, of the indicated indices, such that a multiple of the character χ of interest is a difference of the corresponding permutation characters: $c\chi = \phi_m - \phi_k$. Suppose one has the complete list of all degree m fields corresponding to the permutation representation of G on G/H_m and root discriminant $\leq B$. Then one can extract the complete list of $\mathcal{L}(G, \chi; B^{m/(m-k)})$ of desired Artin L-functions.

For example, consider χ_5, the absolutely irreducible 5-dimensional character of A_6. The permutation character for a corresponding sextic field decomposes $\phi_6 = 1 + \chi_5$, and so the discriminant of the sextic field equals the conductor of χ_5. As an example with $k > 1$, consider the 6-dimension character χ_6 for $C_3 \wr C_3 = 9T17$, which is the sum of a three-dimensional character and its conjugate. The nonic field has a cubic subfield, and the characters are related by $\phi_9 = \phi_3 + \chi_6$. In terms of conductors, $D_9 = D_3 \cdot D_{\chi_6}$, where D_9 and D_3 are field discriminants. So, we can determine the minimal conductor of an L-function with type $(C_3 \wr C_3, \chi_6)$ from a sufficiently long complete list of nonic number fields with Galois group $C_3 \wr C_3$.

This method, applied to both old and newer lists presented in [JR14a], accounts for all but one of the δ_1 reported in Roman type on the same line as a 0 in the # column. The remaining case of an established δ_1 is for the type $(\mathrm{GL}_3(2), \chi_7)$. The group $\mathrm{GL}_3(2)$ appears on our tables as $7T5$. The permutation representation $8T37$ has character $\chi_7 + 1$. Here the general method says that $\mathcal{L}(\mathrm{GL}_3(2), \chi_7; 26.12)$ is empty. It is prohibitive to compute the first octic discriminant by searching among octic polynomials. In [JR] we carried out a long search of septic polynomials, examining all local possibilities giving an octic discriminant at most 30. This computation shows that $|\mathcal{L}(\mathrm{GL}_3(2), \chi_7; 48.76)| = 25$ and in particular identifies $\delta_1 = 21^{8/7} \approx 32.44$.

The complete lists of Galois fields for a group first appearing in degree m were likewise computed by searching polynomials in degree m, targeting for small δ_{Gal}. This single search can give many first root conductors at once. For example, the largest groups on our octic and nonic tables are $S_4 \wr S_2 = 8T47$ and $S_3 \wr S_3 = 9T31$. In these cases, minimal root conductors were obtained for 5 of the 10 and 7 of the 12 faithful χ respectively. Searches adapted to a particular character χ as in the previous paragraph can be viewed as a refinement of our method, with targeting being not for small δ_{Gal} but instead for small δ_χ. Many of the italicized entries in the column δ_1 seem improvable to known minimal root conductors via this refinement.

9.3. **The ratio $\delta_1/\mathfrak{d}$.** In all cases on the table, $\delta_1 > \mathfrak{d}$. Thus, as expected, we did not encounter a contradiction to the Artin conjecture or the Riemann hypothesis. In some cases on the table, the ratio $\delta_1/\mathfrak{d}$ is quite close to 1. As two series of examples, consider S_m with its reflection character $\chi_{m-1} = \phi_m - 1$, and D_m and the sum χ of all its faithful 2-dimensional characters. Then these ratios are as follows:

m	2	3	4	5	6	7	8	9
$\delta_1/\mathfrak{d}$ for (S_m, χ_{m-1})	1.00	1.02	1.2	1.005	1.1	1.007		
$\delta_1/\mathfrak{d}$ for (D_m, χ)			1.09	1.02	1.23	1.006	1.07	1.45

In the cases with the smallest ratios, the transition from no L-functions to many L-functions is commonly abrupt. For example, in the case (S_7, χ_6) the lower bound is $\mathfrak{d} \approx 7.50$ and the first seven rounded root conductors are 7.55, 7.60, 7.61, 7.62, 7.64, 7.66, and 7.66.

When the translation from no L-functions to many L-functions is not abrupt, but there is an L-function with outlyingly small conductor, again $\delta_1/\mathfrak{d}$ may be quite close to 1. As an example, for $(8T25, \chi_7)$, one has $\mathfrak{d} \approx 16.10$ and $\delta_1 = 29^{6/7} \approx 17.93$ yielding $\delta_1/\mathfrak{d} \approx 1.11$. However in this case the next root conductor is $\delta_2 = 113^{6/7} \approx 57.52$, yielding $\delta_2/\mathfrak{d} \approx 3.57$. Thus the close agreement is entirely dependent on the L-function with outlyingly small conductor. Even the second root conductor is somewhat of an outlier as the next three conductors are 71.70, 76.39, and 76.39, so that already $\delta_3/\mathfrak{d} \approx 4.45$.

There are many (G, χ) on the table for which the ratio $\delta_1/\mathfrak{d}$ is around 2 or 3. There is some room for improvement in our analytic lower bounds, for example changing the test function (3.1), varying ϕ over all of P_G, or replacing the exponent $\underline{\widehat{\alpha}}$ with the best possible exponent b. However examples like the one in the previous paragraph suggest to us that in many cases the resulting increase in $\mathfrak{d}$ towards δ_1 would be very small.

9.4. **Multiply minimal fields.** Tables 8.1–8.8 make implicit reference to many Galois number fields, and all necessary complete lists are accessible on the database [JR14a]. Table 9.1 presents a small excerpt from this database by giving six polynomials $f(x)$. For each $f(x)$, Table 9.1 first gives the Galois group G and the root discriminant δ of the splitting field K^{gal}. We are highlighting these particular Galois number fields K^{gal} here because they are *multiply minimal*: they each give rise to the minimal root conductor for at least two different rationally irreducible characters χ. The degrees of these characters are given in the last column of Table 9.1.

G	δ			Polynomial	$\chi(e)$
A_5	$2^{3/2}17^{2/3}$	$\approx$	18.70	$x^5 - x^4 + 2x^2 - 2x + 2$	$4, 6$
A_6	$2^{13/6}3^{16/9}$	$\approx$	31.66	$x^6 - 3x^4 - 12x^3 - 9x^2 + 1$	$9, 10, 16$
S_6	$11^{1/2}41^{2/3}$	$\approx$	39.44	$x^6 - x^5 - 4x^4 + 6x^3 - 6x + 5$	$5, 10$
$\mathrm{SL}_2(3)$	$163^{2/3}$	$\approx$	29.83	$x^8 + 9x^6 + 23x^4 + 14x^2 + 1$	$2, 4$
$8T47$	$2^{31/24}5^{1/2}41^{1/2}$	$\approx$	35.05	$x^8 - 2x^7 + 6x^6 - 2x^5 + 26x^4$ $-24x^3 - 24x^2 + 16x + 4$	$12, 18$
$9T19$	$3^{37/24}7^{3/4}$	$\approx$	23.41	$x^9 - 3x^8 - 3x^7 + 12x^6 - 21x^5$ $+36x^4 - 48x^3 + 45x^2 - 24x + 7$	$8, 8$

Table 9.1. Invariants and defining polynomials for Galois number fields giving rise to minimal root discriminants for at least two rationally irreducible characters χ

Further information on the characters χ is given in Tables 8.1–8.8. An interesting point, evident from repeated 1's in the G-block on these tables, is that five of the six fields K^{gal} are also first on the list of G fields ordered by root discriminant. The exception is the S_6 field on Table 9.1, which is only sixth on the list of Galois S_6 fields ordered by root discriminant.

10. Lower bounds in large degrees

In this section, we continue our practice of assuming the Artin conjecture and Riemann hypothesis for the relevant L-functions. For n a positive integer, let $\Delta_1(n)$ be the smallest root discriminant of a degree n field. As illustrated by Figure 3.1, one has,

$$\liminf_{n\to\infty} \Delta_1(n) \geq \Omega \approx 44.7632. \tag{10.1}$$

Now let $\delta_1(n)$ be the smallest root conductor of an absolutely irreducible degree n Artin representation. Theorem 4.2 of [PM11] uses the quadratic method to conclude that $\delta_1(n) \geq 6.59e^{(-13278.42/n)^2}$. If one repeats the argument there without concerns for effectivity, one gets

$$\liminf_{n\to\infty} \delta_1(n) \geq \sqrt{\Omega} \approx 6.6905. \tag{10.2}$$

The contrast between (10.1) and (10.2) is striking, and raises the question of whether $\sqrt{\Omega}$ in (10.2) can be increased at least part way to Ω.

10.1. **The constant Ω as a limiting lower bound.** The next corollary makes use of the extreme character values $\check{\chi}$ and $\hat{\chi}$ introduced at the beginning of Section 5. It shows that if one restricts the type, then one can indeed increase $\sqrt{\Omega}$ all the way to Ω. We formulate the corollary in the context of rationally irreducible characters, to stay in the main context we have set up. However via (2.4), it can be translated to a statement about absolutely irreducible characters.

Corollary 10.1. *Let (G_k, χ_k) be a sequence of rationally irreducible Galois types of degree $n_k = \chi_k(e)$. Suppose that the number of irreducible constituents (χ_k, χ_k) is bounded, $n_k \to \infty$, and either*

- **A:** $\check{\chi}_k/n_k \to 0$, *or*
- **B:** $\hat{\chi}_k/n_k \to 0$.

Then, assuming the Artin conjecture and Riemann hypothesis for relevant L-functions,

$$\liminf_{k\to\infty} \delta_1(G_k, \chi_k) \geq \Omega. \tag{10.3}$$

Proof. For Case A, Theorem 4.2 using a linear auxiliary character as in (5.1) says

$$\delta_1(G_k, \chi_k) \geq M\left(\frac{n_k}{\check{\chi}_k} + 1, \frac{r_k}{\check{\chi}_k} + 1, (\chi_k, \chi_k)\right)^{1+\check{\chi}_k/n_k}.$$

For Case B, Theorem 4.2 using a Galois auxiliary character as in (5.4) says

$$\delta_1(G_k, \chi_k) \geq M\left(|G_k|, 0, (\chi_k, \chi_k)\right)^{1-\hat{\chi}/n_k}.$$

In both cases, the first argument of M tends to infinity, the second argument does not matter, the third argument does not matter either by boundedness, and the exponent tends to 1. By (3.2), these right sides thus have an infimum limit of at least Ω, giving the conclusion (10.3). □

For the proof of Case B, the square auxiliary character would work equally well through (5.2). Also (10.1), (10.2), and Corollary 10.1 could all be strengthened by considering the placement of complex conjugation. For example, when restricting to the totally real case $c = e$, the Ω's in (10.1), (10.2), and (10.3) are simply replaced by $\Theta \approx 215.3325$.

10.2. **Four contrasting examples.** Many natural sequences of types are covered by either Hypothesis A or Hypothesis B, but some are not. Table 10.1 summarizes four sequences which we discuss together with some related sequences next.

G_k	χ_k	$\check{\chi}_k$	$\widehat{\chi}_k$	n	$\lvert G\rvert$	A	B
$\mathrm{PGL}_2(k)$	Steinberg	1	1	k	k^3-k	✓	✓
S_k	Reflection	1	$k-3$	$k-1$	$k!$	✓	
2_ϵ^{1+2k}	Spin	2^k	0	2^k	2^{1+2k}		✓
$2^k.S_k$	Reflection	k	$k-2$	k	$2^k k!$		

TABLE 10.1. Four sequences of types, with Corollary 10.1 applicable to the first three.

10.2.1. *The group* $\mathrm{PGL}_2(k)$ *and its characters of degree* $k-1$*,* k*, and* $k+1$*.* In the sequence $(\mathrm{PGL}_2(k), \chi_k)$ from the first line of Table 10.1, the index k is restricted to be a prime power. The permutation character ϕ_{k+1} arising from the natural action of $\mathrm{PGL}_2(k)$ on $\mathbb{P}^1(\mathbb{F}_k)$ decomposes as $1+\chi_k$ where χ_k is the Steinberg character. Table 10.1 says that the ratios $\check{\chi}_k/n_k$ and $\widehat{\chi}_k/n_k$ are both $1/k$, so Corollary 10.1 applies through both Hypotheses A and B.

The conductor of χ_k is the absolute discriminant of the degree $k+1$ number field with character ϕ_{k+1}. Thus, in this instance, (10.3) is already implied by the classical (10.1). However, the other nonabelian irreducible characters χ of $\mathrm{PGL}_2(k)$ behave very similarly to χ_k. Their dimensions are in $\{k-1, k, k+1\}$ and their values besides $\chi(e)$ are all in $[-2, 2]$. So suppose for each k, an arbitrary nonabelian rationally irreducible character χ_k of $\mathrm{PGL}_2(k)$ were chosen, in such a way that the sequence (χ_k, χ_k) is bounded. Then Corollary 10.1 would again apply through both Hypotheses A and B. But now the χ_k are not particularly closely related to permutation characters.

10.2.2. *The group* S_k *and its canonical characters.* As with the last example, the permutation character ϕ_k arising from the natural action of S_k on $\{1, \dots, k\}$ decomposes as $1+\chi_k$ where χ_k is the reflection character with degree $k-1$. The second line of Table 10.1 shows that Corollary 10.1 applies through Hypothesis A. In fact, using the linear auxiliary character underlying Hypothesis A here is essential; the limiting lower bound coming from the square or quadratic auxiliary characters is $\sqrt{\Omega}$, and this lower bound is just 1 from the Galois auxiliary character.

Again in parallel to the previous example, the familiar sequence (S_k, χ_k) of types needs to be modified to make it a good illustration of the applicability of Corollary 10.1. Characters of S_k are most commonly indexed by partitions of k, with $\chi_{(k)} = 1$, $\chi_{(k-1,1)}$ being the reflection character, and $\chi_{(1,1,\dots,1,1)}$ being the sign character. However an alternative convention is to include explicit reference to the degree k and then omit the largest part of the partition, so that the above three characters have the alternative names $\chi_{k,()}$, $\chi_{k,(1)}$, and $\chi_{k,(1,\dots,1,1)}$. With this convention, one can prove that for any fixed partition μ of a positive integer m, the sequence of types $(G_k, \chi_{k,\mu})$ satisfies Hypothesis A but not B.

The case of general μ is well represented by the two cases where $m = 2$. In these two cases, information in the same format as Table (10.1) is

G_k	χ_k	$\check{\chi}_k$	$\widehat{\chi}_k$	n
S_k	$\chi_{k,(1,1)}$	$\lfloor \frac{k-1}{2} \rfloor$	$\frac{1}{2}(k-2)(k-5)$	$\frac{1}{2}(k-1)(k-2)$
S_k	$\chi_{k,(2)}$	1	$\frac{1}{2}(k-2)(k-5)+1$	$\frac{1}{2}(k-1)(k-2)-1$

.

Let $X_{k,m}$ be the S_k-set consisting of m-tuples of distinct elements of $\{1, \ldots, k\}$. Then its permutation character $\phi_{k,m}$ decomposes into $\chi_{k,\mu}$ with μ a partition of an integer $\leq m$. These formulas are uniform in k, as in

$$\phi_{k,2} = \chi_{k,(1,1)} + \chi_{k,(2)} + 2\chi_{k,(1)} + \chi_{k,()}.$$

For μ running over partitions of a large integer m, the characters $\chi_{k,\mu}$ can be reasonably regarded as quite far from permutation characters, and they thus serve as a better illustration of Corollary 10.1. The sequences $(S_k, \chi_{k,\mu})$ satisfy Hypothesis A but not B, because n_k and $\widehat{\chi}_k$ grow polynomially as k^m, while $\check{\chi}_k$ grows polynomially with degree $< m$.

10.2.3. *The extra-special group 2_ϵ^{1+2k} and its degree 2^k character.* Fix $\epsilon \in \{+, -\}$. Let G_k be the extra-special 2-group of type ϵ and order 2^{1+2k}, so that 2_+^{1+2} and 2_-^{1+2} are the dihedral and quaternion groups respectively. These groups each have exactly one irreducible character of degree larger than 1, this degree being 2^k. There are just three character values, -2^k, 0, and 2^k. For these two sequences, Corollary 10.1 again applies, but now only through Hypothesis B.

10.2.4. *The Weyl group $2^k.S_k$ and its degree k reflection character.* The Weyl group $W(B_k) \cong 2^k.S_k$ of signed permutation matrices comes with its defining degree k character χ_k. Here, as indicated by the fourth line of Table 10.1, neither hypothesis of Corollary 10.1 applies.

However the conclusion (10.3) of Corollary 10.1 continues to hold as follows. Relate the character χ_k in question to the two standard permutation characters of $2^k.S_k$ via $\phi_{2k} = \phi_k + \chi_k$. For a given $2^k.S_k$ field, $D_{\Phi_{2k}} = D_{\Phi_k} D_{\mathcal{X}_k}$. But, since Φ_k corresponds to an index 2 subfield of the degree $2k$ number field for Φ_{2k}, we have $D_{\Phi_k}^2 \mid D_{\Phi_{2k}}$. Combining these we get $D_{\Phi_k} \mid D_{\mathcal{X}_k}$ and hence $\delta_{\Phi_k} < \delta_{\mathcal{X}_k}$. So (10.1) implies (10.3).

10.3. **Concluding speculation.** As we have illustrated in §10.2.1–10.2.3, both Hypothesis A and Hypothesis B are quite broad. This breadth, together with the fact that the conclusion (10.3) still holds for our last sequence, raises the question of whether (10.3) can be formulated more universally. While the evidence is far from definitive, we expect a positive answer. Thus we expect that the first accumulation point of the numbers $\delta_1(G, \chi)$ is at least Ω, where (G, χ) runs over all types with χ irreducible. Phrased differently, we expect that the first accumulation point of the root conductors of all irreducible Artin L-functions is at least Ω.

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