COMPUTING SEPTIC NUMBER FIELDS

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Abstract. We determine all degree 7 number fields where the absolute value of the discriminant is $\leq 2 \cdot 10^8$. To accomplish this large-scale computation, we distributed the computation over many servers on the internet through the BOINC network.

For a number field K, let D_K denote its discriminant and r_K its number of real embeddings. A number of papers determine sets

$$\mathcal{N}(n, r, B) = \{K \mid n = [K : \mathbb{Q}], r_K = r, \text{ and } |D_K| \leq B\}$$

where number fields are considered up to isomorphism. Computing sets $\mathcal{N}(n, r, B)$ for different combinations of n, r, and B has received a sizable amount of attention from mathematicians. The resulting lists have been applied by number theorists for a variety of purposes, such as in the study of Diophantine equations [DKMS17], modular forms and their Galois representations [Buz14, DDR16], and in the arithmetic statistics of number fields [Rob01, CT16].

One case of computing sets $\mathcal{N}(n, r, B)$ which has relatively little presence in the literature is when the degree $n = 7$. Several early papers found the fields with the smallest discriminant for each signature r_K [DyD83, DyD84, DyD88, Poh77]. The most extensive computations in the totally real case of $r_K = 7$ are by Voight [Voi] which extend his results from [Voi08]. In all other signatures, the most extensive computations in the literature go back to the thesis of Létard [Lét95]. Together, these computations result in a relatively small number of fields; 495 fields with $r_K < 7$ and 837 fields with $r_K = 7$. For many applications, such as initial explorations in the arithmetic statistics of septic number fields, longer complete lists of fields are necessary. However, more extensive computations of this type are limited by the computation time required.

Here, we extend previous results significantly for signatures $r_K < 7$ yielding over a million number fields. Section 1 states the main theorem and provides data on the resulting fields. Section 2 describes the method used to carry out the computation, with an emphasis on the mechanics of parallelization which made such a large computation feasible.

1. Results

Our main result is the following.

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Theorem 1. *Up to isomorphism, there are* 1,009,140 *septic number fields with* $|D_K| \leq 2 \cdot 10^8$.

The defining polynomials themselves can be accessed from the web site [JR], and will soon be added to [LMF18].

Our computation did not distinguish signatures, and so it overlapped with part of the search of [Voi], acting as a check on both the data from that site and on the implementation used here. The results were consistent with the known results from both [Voi] and [Lét95].

Table 1 shows the breakdown of the number of fields found by Galois group and signature. Here G is the Galois group of the polynomial, which must be one of the seven possibilities shown. For each combination of group and signature, the smallest

$r_K \backslash G$	C_7	D_7	$C_7 : C_3$	F_7	$\mathrm{GL}_3(2)$	A_7	S_7	Total
1	–	18	–	9	–	–	666,205	666,232
3	–	–	–	–	192	115	319,896	320,203
5	–	–	–	–	–	–	22,432	22,432
7	0	1	0	0	0	0	272	273
Total	0	19	0	9	192	115	1,008,805	1,009,140

Table 1. Numbers of septic fields K with given Galois groups and signatures with $|D_K| \leq 2 \cdot 10^8$. Dashes represent combinations which are not possible for any discriminant bound.

discriminant field was previously known [KM01, Thm. 12]. Thus, the five entries of 0 in Table 1 were expected.

Since non-isomorphic fields with the same discriminant is a relatively uncommon phenomenon, we counted repetitions. The discriminant $-158{,}210{,}496$ appears six times, the discriminant $-160{,}823{,}232$ appears five times, 11 discriminants appear four times, 164 discriminants appear three times, and 7522 disciminants appear twice.

2. Methods

The computation employed a standard algorithm known as a Hunter search, as explained in [Coh00, §9.3]. It searches for monic defining polynomials with integer coefficients via seven nested loops, one for each unknown coefficient of the polynomial. Bounds for each coefficient come from the geometry of numbers and depend on previously determined coefficients, i.e., those in outer loops. Once all of the coefficients are determined, the polynomial is checked for irreducibility, and to see if its absolute discriminant fits within our bound.

This approach was too time consuming to complete on the computer(s) directly available to the authors. To work around this, we used the BOINC network [BOI], a system originally created to support the SETI@home project. People on the internet sign up for BOINC projects and contribute computing resources on their computers when the computer is otherwise idle.

A Hunter search is easily parallelized due to its structure of simple nested loops. We locally ran a setup phase which executed the five outermost loops of the search. For each combination of the five corresponding coefficients, we save to disk the vector of coefficients and the information needed for the two innermost loops. There were roughly 267 million vectors of coefficients.

Next, the list of vectors of coefficients was put in random order, and then separated them into groups of 89 per work unit, giving approximately 3 million work units to be sent to BOINC clients. The vectors were randomized before grouping to even the computational time of the work units.

After roughly 4 months of real time, the network clients combined for roughly 3780 cpu-years of computing time. Roughly 30,000 computers took part in the computation. Altogether, these clients inspected 38,136,713,961,009 polynomials, which led to the million fields in Theorem 1.

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