p-adic heights for semi-stable abelian varieties

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In [9], P. Schneider defines an analytic p-adic height pairing for semi-stable ordinary abelian varieties. In [5], Mazur and Tate show that Schneider's analytic height is equivalent to their canonical height defined via biextensions. If the abelian variety has good ordinary reduction, Schneider defines a second (algebraic) height pairing in [10] with which he proves Birch and Swinnerton-Dyer type results for certain Iwasawa L-functions. In [11], he shows that his two heights are the same up to sign.

In this paper, we will define an algebraic height for semi-stable ordinary abelian varieties and show that it is the same (up to sign) as Schneider's analytic height. In [2], we will use our height to investigate Birch and Swinnerton-Dyer properties for Iwasawa L-functions of abelian varieties with exceptional zeros.

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1 Notations, Conventions and Terminology

We will observe the following notations, conventions and terminology throughout:

Let $p \neq 2$ be a prime number and let K be a number field. If F is a field, $\mathcal{O}_F$ will be its ring of integers (where appropriate).

A *quasi-isomorphism* is a group homomorphism with finite kernel and cokernel. Similarly, we will refer to *quasi-injections*, *quasi-surjections*, and *quasi-exact sequences* with the obvious meanings.

If G is an abelian group, we let G^* denote the Pontryagin dual; $\operatorname{Tor} G :=$ the torsion subgroup of G and $G_{\mathrm{Tor}} := G/\operatorname{Tor} G$. Furthermore, if $f\colon A \to B$ is a homomorphism, we let $\operatorname{Tor} f$ and f_{Tor} be the induced maps $\operatorname{Tor} A \to \operatorname{Tor} B$ and $A_{\mathrm{Tor}} \to B_{\mathrm{Tor}}$ respectively. We will use simply G to denote the order of G.

If G is an abelian group, an abelian group scheme or a sheaf of abelian groups, G_n will denote the kernel of multiplication by n on G. We then let $G_{p^\infty} := \varinjlim G_{p^m}$ and $T_p(G) := \varprojlim G_{p^m}$.

The additive topological group $\mathbf{Z}_p$ will be denoted by Γ. The closed subgroup of index p^n will be written Γ_n.

Let L be a Γ-extension of K (i.e., an extension such that $Gal(L/K) \cong \Gamma$). We denote the fixed field of Γ_n by K_n. An example of a Γ-extension is the *cyclotomic* extension of K, the unique Γ-extension contained in $K(\boldsymbol{\mu}_{p^\infty})$, (the field gotten by adjoining all of the p^{th}-power roots of unity to K). Corresponding to L/K is a p-adic logarithm denoted $\log_p$. If L is the cyclotomic extension, then $\log_p$ is Iwasawa's logarithm.

If t is a place of K, we will denote the completion of K at t by K_t. If L is a Γ-extension of K and t is a place which ramifies in L, then we define L_t to be the compositum of L and K_t. We write Γ_t for $Gal(L_t/K_t)$. The set of all the primes of K ramifying in L will be denoted by T. The places of L lying above T will be denoted by T_∞.

Fix a continuous character

$$\kappa\colon \mathrm{Gal}(L/K) \to 1 + p^e\mathbf{Z}_p \subset \mathbf{Z}_p^*$$

such that the following diagram is commutative

$$\begin{array}{ccc} K_v^* & \xrightarrow{\text{rec}} & \Gamma \\ \Big\downarrow \log_p & & \Big\downarrow \kappa \\ \mathbf{Z}_p & \xleftarrow{\log_p} & \mathbf{Z}_p^* \end{array}$$

If L/K is the cyclotomic extension, then κ is the cyclotomic character (see [9] p. 402).

Let $A_{/K}$ be an abelian variety over K, and A its Neron model over $\mathcal{O}_K$. The Neron model of the dual abelian variety of A is denoted $\tilde{A}$.

If E/K is a finite Galois extension of fields, then we denote the norms of the points of A by

$$\mathrm{Norm}_{E/K} A := \bigcup_{x\in A(E)} \sum_{g\in \mathrm{Gal}(E/K)} g\, x$$

where the sum is taken under the group law of G.

The *universal norms* of A for L/K is the intersection of the groups of norms from each finite subextension E

$$NA(K) := \bigcap_{K\subset E\subset L} \mathrm{Norm}_{E/K} A.$$

We let A^0 be the "connected component" of A. One then has the short exact sequence of sheaves for the *fpqf* site

$$0 \to A^0 \to A \to \Phi \to 0.$$

The restriction of Φ to the small etale site is a skyscraper sheaf composed on each fiber by the group of connected components for the reduction of A at that place. That is,

$$\Phi = \bigoplus_{v\in \mathcal{O}_K} {}_v\Phi$$

where all but finitely many of the ${}_v\Phi$ are trivial. The *fpqf* cohomology of Φ can be computed in the etale site by a theorem of Grothendieck (see [1]) since it is the quotient of smooth group schemes.

The image of multiplication by powers of p in A will stabilize for high powers. It is denoted by $\bar{A}$. If ν is sufficiently large and A is semi-stable at the primes of K dividing p, there are short exact sequences for the *fpqf* site

$$0 \to \bar{A} \to A \to \Phi/p^\nu\Phi \to 0$$

and

$$0 \to A_{p^\nu} \to A \xrightarrow{p^\nu} \bar{A} \to 0. \tag{1}$$

For each of these constructions (A, A_{p^∞}, A^0, $\bar{A}$, Φ) we will not usually explicitly state to what base field they correspond, as it should be clear from the context. In the case of A^0, there is little confusion possible since A^0 base changes properly if A is semi-stable.

We put the following conditions on A, L and K.

1. the reduction of A is semi-stable at every place of K dividing p, and is an extension of an ordinary abelian variety by a torus for every $t \in T$;

2. for every place $t \in T$, the universal norms of $A(L_t)$ in $A(K_t)$ are of finite index.

The second condition is known to hold in the following situations

- A has good ordinary reduction at v (see [3]);

- A is an elliptic curve with non-split multiplicative reduction at v (see [8]);

- A is an elliptic curve with split multiplicative reduction at v and $\log_p q \neq 0$, where $\log_p$ is a logarithm associated to the Γ-extension L/K, and q is the Tate parameter of A at v. This is, conjecturally, always the case for the cyclotomic extension (see [6]).

We will be making frequent use of Galois cohomology and flat cohomology. For Galois cohomology groups we will use the standard notation $H^i(G, F)$, where G is the Galois group and F a G module. If G is infinite, we mean cohomology with continuous cochains, and $H^i(\ /K, F)$ denotes $H^i(Gal(\overline{K}/K), F)$. By $\mathrm{cd}_p(G) = n$, we mean that $H^i(G, F) = 0$ for all $i > n$ and p-torsion G-modules F.

Other cohomology groups will be taken for the (big) *fpqf* site, unless otherwise noted. The category of abelian sheaves for this site over a scheme X will be denoted by $Sh(X)$. If $X = spec(R)$, then we will write $Sh(R)$. Similarly, if R is a ring, $H^i(spec(R), F)$ will be denoted simply by $H^i(R, F)$.

If R is a local ring, then we denote the relative cohomology over $spec(R)$ with values in a sheaf F by $H^i_.(R, F)$.

We will be using equivariant cohomology. See the appendix to [10] for its definitions and basic properties. We will use the same notation as [10] (i.e., $H^i(\mathcal{O}_F/\mathcal{O}_K, \mathcal{F})$ denotes the equivariant cohomology of the sheaf $\mathcal{F} \in Sh(F)$ for a Galois extension F/K).

2 Selmer Groups

Selmer groups play a central role in our definition of the algebraic p-adic height pairing. The classical p-Selmer group of an abelian variety $A_{/K}$, $\mathbf{S}_{clas}$, is defined in terms of Galois cohomology as the group which makes the following diagram exact and commute:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A(K) \otimes \mathbf{Q}_p/\mathbf{Z}_p & \to & \mathbf{S}_{clas} & \to & III_{p^\infty} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A(K) \otimes \mathbf{Q}_p/\mathbf{Z}_p & \to & H^1(\ /K, A_{p^\infty}(\overline{K})) & \to & H^1(\ /K, A(\overline{K}))_{p^\infty} & \to & 0 \end{array}$$

Here, III is the Shafarevich-Tate group for $A_{/K}$ defined as the intersection of the kernels $H^1(\ /K, A(\overline{K})) \to H^1(\ /K_v, A(\overline{K_v}))$ where v runs over all of the places of K.

The Selmer group can be interpreted in terms of flat cohomology groups as follows.

Proposition 2.1 $\mathbf{S}_{clas}$ *is canonically isomorphic to the image of* $H^1(\mathcal{O}_K, A^0_{p^\infty})$ *in* $H^1(\mathcal{O}_K, A_{p^\infty})$.

Proof: Use Kummer sequences and the fact that the image of $H^1(\mathcal{O}_K, A^0)_{p^\infty}$ in $H^1(\ /K, A)_{p^\infty}$ is isomorphic to the p-primary component of III_K (see appendix to [3]). Here we use that $p \neq 2$. □

Remark: If S is a set of places of K which contain all the places of bad reduction for A, then since $H^1_S(\mathcal{O}_K, A_{p^\infty}) = 0$ (see the proof of Lemma 3.1), the long exact sequence in relative cohomology shows that $H^1(\mathcal{O}_K, A_{p^\infty})$ injects into $H^1(\mathcal{O}_K - S, A_{p^\infty})$. We can therefore identify $\mathbf{S}_{clas}$ with the image of

$$H^1(\mathcal{O}_K, A^0_{p^\infty}) \quad \text{in} \quad H^1(\mathcal{O}_K - S, A_{p^\infty}).$$

Note that since we have included all of the places of bad reduction in S,

$$H^1(\mathcal{O}_K - S, A^0_{p^\infty}) = H^1(\mathcal{O}_K - S, A_{p^\infty}).$$

Motivated by the above remark, we will use the groups

$$\mathcal{S}_n := \text{Image } H^1(\mathcal{O}_{K_n}, A^0_{p^\infty}) \text{ in } H^1(\mathcal{O}_{K_n} - T, A^0_{p^\infty})$$

and their direct limit $\mathcal{S}_\infty$ (we also write $\mathcal{S}$ for $\mathcal{S}_0$). These groups differ from the classical Selmer groups by local cohomology groups at places of bad reduction which are unramified in L. If these places split finitely in L, these local groups have orders bounded independently of n. In this case, $\mathcal{S}_\infty$ is a Λ-module quasi-isomorphic to the Selmer group over L.

3 Defining the Height Pairing

We now define a p-adic height pairing which generalizes Schneider's algebraic height in [10]. In section 4.2, we will show that it is the same, up to sign, as Schneider's analytic height. We write $H^1(\mathcal{O}_K, T_p(A))$ for $\varprojlim H^1(\mathcal{O}_K, A_{p^\nu})$, and similarly for $\tilde{A}$. The pairing,

$$\langle\langle\ ,\ \rangle\rangle_\gamma\colon H^1(\mathcal{O}_K, T_p(\tilde{A})) \times H^1(\mathcal{O}_K, T_p(A)) \to \mathbf{Q}_p$$

is defined by the following diagram.

$$\begin{tikzcd}
H^1(\mathcal{O}_K, T_p(\tilde{A})) \arrow[d, equal] & \mathrm{Hom}_{\mathbf{Z}_p}(H^1(\mathcal{O}_K, T_p(A)), \mathbf{Z}_p) \arrow[d, equal] \\
H^2(\mathcal{O}_K, A_{p^\infty})^* \arrow[d, "\varphi_2^*"] & (H^1(\mathcal{O}_K, A^0_{p^\infty})^*)_{\mathrm{Tor}} \\
H^2(\mathcal{O}_K, A^0_{p^\infty})^* & H^1(\mathcal{O}_K, A^0_{p^\infty})^* \arrow[u] \\
{[H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})/H^1_T(\mathcal{O}_L/\mathcal{O}_K)]^*} \arrow[u, "\theta^*"] \arrow[d, "\mu^*"] & \mathcal{S}^* \arrow[u, "i^*"] \\
{[H^1(\mathcal{O}_L, A^0_{p^\infty})_\Gamma/H^1_T(\mathcal{O}_L)]^*} \arrow[d, equal] & \\
H^1(\Gamma, \mathcal{S}_\infty)^* \arrow[r, "g^*"] & H^0(\Gamma, \mathcal{S}_\infty)^* \arrow[uu, "\alpha^*"]
\end{tikzcd} \tag{2}$$

Explanations: We will use the notation $H^1_T(\mathcal{O}_L/\mathcal{O}_K)$ and $H^1_T(\mathcal{O}_L)$ to denote the image of the group $H^1_{T_\infty}(\mathcal{O}_L, A^0_{p^\infty})_\Gamma$ in $H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})$ and $H^1(\mathcal{O}_L, A^0_{p^\infty})_\Gamma$ respectively. We now look at the maps used above.

Starting at the top of the left column, we note that

$$\varprojlim H^1(\mathcal{O}_K, \tilde{A}_{p^\nu}) = H^2(\mathcal{O}_K, A_{p^\infty})^*$$

by McCallum's global duality ([7], Theorem 5.3)

$$H^i(\mathcal{O}_K, \tilde{A}_{p^\nu}) = H^{3-i}(\mathcal{O}_K, A_{p^\nu})^*.$$

The map φ_2 is a special case of the maps

$$\varphi_i\colon H^i(\mathcal{O}_K, A^0_{p^\infty}) \to H^i(\mathcal{O}_K, A_{p^\infty})$$

which are quasi-isomorphisms.

For the map θ, we start with $H^2(\mathcal{O}_K, A^0_{p^\infty}) \to H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})$. (There is a discussion of the definition of maps such as this in the following section.) We then compose with the projection to $H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})/H^1_T(\mathcal{O}_L/\mathcal{O}_K)$ to get θ. It is a quasi-isomorphism by Proposition 3.6 below.

We define the map

$$\mu\colon H^1(\mathcal{O}_L, A^0_{p^\infty})_\Gamma/H^1_T(\mathcal{O}_L)_\Gamma \to H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})/H^1_T(\mathcal{O}_L/\mathcal{O}_K)$$

from the following diagram.

$$\begin{array}{ccccccccc}
 & & H^1_{T_\infty}(\mathcal{O}_L, A^0_{p^\infty})_\Gamma & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \to & H^1(\mathcal{O}_L, A^0_{p^\infty})_\Gamma & \to & H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) & \to & H^2(\mathcal{O}_L, A^0_{p^\infty})^\Gamma & \to & 0
\end{array}$$

Clearly, μ is injective, and the obstruction to it being surjective is $H^2(\mathcal{O}_L, A^0_{p^\infty})^\Gamma$.

The last equality in the first column comes from

$$H^1_{T_\infty}(\mathcal{O}_L, A^0_{p^\infty}) \to H^1(\mathcal{O}_L, A^0_{p^\infty}) \twoheadrightarrow \mathcal{S}_\infty.$$

First we break it into two short exact sequences

$$0 \to X \to H^1_{T_\infty}(\mathcal{O}_L, A^0_{p^\infty}) \to Y \to 0$$

$$0 \to Y \to H^1(\mathcal{O}_L, A^0_{p^\infty}) \to \mathcal{S}_\infty \to 0$$

Then, taking cohomology of Γ and using that $\mathrm{cd}_p(\Gamma) = 1$, we get the sequences

$$X_\Gamma \to H^1_{T_\infty}(\mathcal{O}_L, A^0_{p^\infty})_\Gamma \to Y_\Gamma \to 0$$

and

$$Y_\Gamma \to H^1(\mathcal{O}_L, A^0_{p^\infty})_\Gamma \to (\mathcal{S}_\infty)_\Gamma \to 0$$

which together yield the desired equality.

The map g is induced by the inclusion of $H^0(\Gamma, \mathcal{S}_\infty)$ in $\mathcal{S}_\infty$, followed by the projection to the co-invariants $(\mathcal{S}_\infty)_\Gamma = H^1(\Gamma, \mathcal{S}_\infty)$.

The definition of the map α comes from the definitions of $\mathcal{S}$ and $\mathcal{S}_\infty$. It is a quasi-isomorphism by Proposition 3.7 below.

The map i comes from the identification of $\mathcal{S}$ with the image of $H^1(\mathcal{O}_K, A^0_{p^\infty})$ in $H^1(\mathcal{O}_K - T, A^0_{p^\infty})$. Thus, i is surjective, and its kernel is bounded by the order of $H^1_T(\mathcal{O}_K, A^0_{p^\infty}) = H^0_T(\mathcal{O}_K, \Phi_{p^\infty})$ (by Lemma 3.1) which is finite. Therefore, i is a quasi-isomorphism.

The map $H^1(\mathcal{O}_K, A^0_{p^\infty}) \to (H^1(\mathcal{O}_K, A^0_{p^\infty})^*)_{Tor}$ is the natural projection, and is a quasi-isomorphism because $H^1(\mathcal{O}_K, A^0_{p^\infty})$ is a finitely generated $\mathbf{Z}_p$-module, and hence, its torsion is finite.

Finally, we can identify

$$\mathrm{Hom}_{\mathbf{Z}_p}(H^1(\mathcal{O}_K, T_p(A)), \mathbf{Z}_p) \quad \text{with} \quad (H^1(\mathcal{O}_K, A^0_{p^\infty})^*)_{\mathrm{Tor}}$$

from Schneider ([11], p. 334).

This finishes the definition of the height pairing. Note that (with the propositions in the following section) all of the maps used in the definition are quasi-isomorphisms except for possibly μ^* and g^*, but μ^* is surjective.

3.1 Cohomological Computations

We now prove the propositions alluded to in the above definition. We start with a few preliminaries.

Lemma 3.1 *If v is a place of K, then*

$$H^i_{\cdot}(\mathcal{O}_{K_v}, A^0_{p^\infty}) = \begin{cases} 0 & \text{for } i = 0 \\ \Phi_{p^\infty}(\mathcal{O}_{K_v}) & \text{for } i = 1 \\ (\tilde{A}(K_v)^*)_{p^\infty} & \text{for } i = 2 \\ 0 & \text{for } i > 2. \end{cases}$$

Proof: By definition,

$$H^0_{\cdot}(\mathcal{O}_{K_v}, A^0_{p^\infty}) = \ker\,[H^0(\mathcal{O}_{K_v}, A^0_{p^\infty}) \to H^0(K_v, A^0_{p^\infty})] = 0.$$

Similarly,

$$H^0_{\cdot}(\mathcal{O}_{K_v}, A) = H^0_{\cdot}(\mathcal{O}_{K_v}, \bar{A}) = H^0_{\cdot}(\mathcal{O}_{K_v}, A_{p^\infty}) = 0.$$

For $H^1_{\cdot}(\mathcal{O}_{K_v}, A^0_{p^\infty})$, we begin with the exact sequence

$$0 \to A^0_{p^\infty} \to A_{p^\infty} \to \Phi_{p^\infty} \to 0. \tag{3}$$

From the long exact sequence in relative cohomology we get

$$H^0_{\cdot}(\mathcal{O}_{K_v}, A_{p^\infty}) \to H^0_{\cdot}(\mathcal{O}_{K_v}, \Phi_{p^\infty}) \to H^1_{\cdot}(\mathcal{O}_{K_v}, A^0_{p^\infty}) \to H^1_{\cdot}(\mathcal{O}_{K_v}, A_{p^\infty}).$$

As noted above, $H^0_{\cdot}(\mathcal{O}_{K_v}, A_{p^\infty}) = 0$. It is also true that $H^1_{\cdot}(\mathcal{O}_{K_v}, A_{p^\infty}) = 0$. To see this, note that

$$H^1_{\cdot}(\mathcal{O}_{K_v}, A_{p^\infty}) = \varinjlim H^1_{\cdot}(\mathcal{O}_{K_v}, A_{p^\nu}).$$

Furthermore, we have the exact sequence

$$H^0_{\cdot}(\mathcal{O}_{K_v}, \bar{A}) \to H^1_{\cdot}(\mathcal{O}_{K_v}, A_{p^\nu}) \to H^1_{\cdot}(\mathcal{O}_{K_v}, A)$$

which comes from the short exact sequence

$$0 \to A_{p^\nu} \to A \to \bar{A} \to 0$$

(see equation 1). We know that $H^0_!(\mathcal{O}_{K_v}, \bar{A}) = 0$ and $H^1_!(\mathcal{O}_{K_v}, A) = 0$ (by [7], Proposition 3.2) and therefore, $H^1_!(\mathcal{O}_{K_v}, A_{p^\infty}) = 0$. Hence,

$$H^0_!(\mathcal{O}_{K_v}, \Phi_{p^\infty}) = H^1_!(\mathcal{O}_{K_v}, A^0_{p^\infty}).$$

Then, since the cohomology of Φ_{p^∞} may be computed on the small etale site where it is a skyscraper sheaf, $H^0_!(\mathcal{O}_{K_v}, \Phi_{p^\infty}) = \Phi_{p^\infty}(\mathcal{O}_{K_v})$.

For $i = 2$, we have that $H^2_!(\mathcal{O}_{K_v}, A^0_{p^\infty}) = H^2_!(\mathcal{O}_{K_v}, A^0)_{p^\infty}$ from the Kummer sequence

$$0 \to H^1_!(\mathcal{O}_{K_v}, A^0) \otimes \mathbf{Q}_p/\mathbf{Z}_p \to H^2_!(\mathcal{O}_{K_v}, A^0_{p^\infty}) \to H^2_!(\mathcal{O}_{K_v}, A^0)_{p^\infty} \to 0$$

and the fact that $H^1_!(\mathcal{O}_{K_v}, A^0)$ is finite. The result then follows from local duality (see [7], Theorem 4.3).

We can calculate $H^3_!(\mathcal{O}_{K_v}, A^0_{p^\infty})$, as follows.

$$\begin{aligned} H^3_!(\mathcal{O}_{K_v}, A^0_{p^\infty}) &= \varinjlim H^3_!(\mathcal{O}_{K_v}, A^0_{p^\nu}) \\ &= \varinjlim H^3_!(\mathcal{O}_{K_v}, A_{p^\nu}) \\ &= (\varprojlim \tilde{A}_{p^\nu}(\mathcal{O}_{K_v}))^* \\ &= 0. \end{aligned}$$

The second equality follows from the long exact sequence in cohomology associated to the short exact sequence 3 and the fact that the cohomology of Φ_{p^∞} vanishes in dimensions 2 and higher. (Again, computing its cohomology on the small etale site where it is a skyscraper sheaf. It may then, by [3] p. 222, be computed in terms of Galois cohomology over the residue field, which has p-cohomological dimension one.) The last equality follows because $\tilde{A}_{p^\infty}(K_v)$ is finite.

The higher relative cohomology groups vanish in dimensions greater than three simply by using the Kummer sequence

$$\cdots \to H^{i-1}_!(\mathcal{O}_{K_v}, A^0) \to H^i_!(\mathcal{O}_{K_v}, A^0_{p^\infty}) \to H^i_!(\mathcal{O}_{K_v}, A^0) \to \cdots$$

and the vanishing of $H^i_!(\mathcal{O}_{K_v}, A^0)$ for $i \geq 3$. $\square$

Remark: The above lemma naturally holds for any finite extension $K_{n,v}$ of K_v, as well as to $\mathcal{O}_{L_v}$ for $i \neq 2$ by taking the direct limit.

Lemma 3.2 *The natural map*

$$H^2_T(\mathcal{O}_K, A^0_{p^\infty}) \to H^2_{T_\infty}(\mathcal{O}_L, A^0_{p^\infty})^\Gamma$$

is surjective and its kernel is finite of order

$$\prod_{t \in T} (\tilde{A}(K_t)/N\tilde{A}(K_t))_{p^\infty}.$$

Proof: Let E_T denote ker $[H^2_T(\mathcal{O}_K, A^0_{p^\infty}) \to H^2_{T_\infty}(\mathcal{O}_L, A^0_{p^\infty})^\Gamma]$, and T_n denote the places of $\mathcal{O}_{K_n}$ lying over T. Then,

$$\begin{aligned}
E_T &= \ker\, [H^2_T(\mathcal{O}_K, A^0_{p^\infty}) \to H^2_{T_\infty}(\mathcal{O}_L, A^0_{p^\infty})] \\
&= \varinjlim \ker\, [H^2_T(\mathcal{O}_K, A^0_{p^\infty}) \to H^2_{T_n}(\mathcal{O}_{K_n}, A^0_{p^\infty})] \\
&= \varinjlim \ker\, [\bigoplus_{t\in T} H^2_{\cdot}(\mathcal{O}_{K_t}, A^0_{p^\infty}) \to \bigoplus_{v\in T_n} H^2_{\cdot}(\mathcal{O}_{K_{n,v}}, A^0_{p^\infty})] \\
&= \varinjlim \bigoplus_{t\in T} \ker\, [H^2_{\cdot}(\mathcal{O}_{K_t}, A^0_{p^\infty}) \to H^2_{\cdot}(\mathcal{O}_{K_{n,t}}, A^0_{p^\infty})] \\
&= \bigoplus_{t\in T} \varinjlim \ker\, [H^2_{\cdot}(\mathcal{O}_{K_t}, A^0_{p^\infty}) \to H^2_{\cdot}(\mathcal{O}_{K_{n,t}}, A^0_{p^\infty})] \\
&= \bigoplus_{t\in T} (\varprojlim \operatorname{cok}\, [H^0(\mathcal{O}_{K_{n,t}}, \tilde{A}) \to H^0(\mathcal{O}_{K_t}, \tilde{A})]_{p^\infty})^* \\
&= \bigoplus_{t\in T} (\varprojlim \operatorname{cok}\, [\tilde{A}(K_{n,t}) \to \tilde{A}(K_t)]_{p^\infty})^* \\
&= \bigoplus_{t\in T} ((\tilde{A}(K_t)/N\tilde{A}(K_t))_{p^\infty})^*.
\end{aligned}$$

Here we employed the Kummer sequence for A^0 and local duality (see [7], Theorem 4.3). This group is finite since the cokernel of the local norm mapping is finite for $\tilde{A}$ iff it is for A (which we is one of our assumptions on A).

Now we show surjectivity. Since $H^2_T(\mathcal{O}_K, A^0_{p^\infty}) = H^2_T(\mathcal{O}_K, A^0)_{p^\infty}$ from the Kummer theory of A^0 over $\mathcal{O}_K$, and analogously over $\mathcal{O}_L$, we have

$$\begin{aligned}
\operatorname{cok}\, [H^2_T(\mathcal{O}_K, A^0_{p^\infty}) &\to H^2_{T_\infty}(\mathcal{O}_L, A^0_{p^\infty})^\Gamma] = \\
&= \operatorname{cok}\, [H^2_T(\mathcal{O}_K, A^0)_{p^\infty} \to H^2_{T_\infty}(\mathcal{O}_L, A^0)^\Gamma_{p^\infty}] \\
&\subset \operatorname{cok}\, [H^2_T(\mathcal{O}_K, A^0) \to H^2_{T_\infty}(\mathcal{O}_L, A^0)^\Gamma] \\
&= \varinjlim \operatorname{cok}\, [H^2_T(\mathcal{O}_K, A^0) \to H^2_{T_n}(\mathcal{O}_{K_n}, A^0)^{\Gamma/\Gamma_n}] \\
&= \bigoplus_{t\in T} \varinjlim \operatorname{cok}[H^2_{\cdot}(\mathcal{O}_{K_t}, A^0) \to H^2_{\cdot}(\mathcal{O}_{K_{n,t}}, A^0)^{\Gamma_t/\Gamma_{n,t}}] \\
&= \bigoplus_{t\in T} \varinjlim \operatorname{cok}\, [H^1(K_t, A) \to H^1(K_{n,t}, A)^{\Gamma_t/\Gamma_{n,t}}] \\
&= \bigoplus_{t\in T} \operatorname{cok}\, [H^1(K_t, A) \to H^1(L_t, A)^{\Gamma_t}]
\end{aligned}$$

where the second to last equation comes from the relative cohomology sequence. But, by the Hochshild-Serre spectral sequence, the final cokernel is contained in

$$\bigoplus_{t\in T} H^2(\Gamma_t, H^1(L_t, A)) = 0$$

because $\mathrm{cd}_p(\Gamma_t) = 1$. □

We now consider equivariant cohomology groups which are relevant to the map θ above.

Proposition 3.3

$$H^i_T(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) = 0 \quad \textit{for} \quad i \neq 1 \ \textit{ or } \ 2.$$

For the non-trivial groups, we have that

$$H^1_T(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) = \bigoplus_{t \in T} H^0(\Gamma_t, {}_t\Phi(\mathcal{O}_{L_t})_{p^\infty}),$$

and the short exact sequence

$$0 \to H^1_T(\mathcal{O}_L, A^0_{p^\infty})_\Gamma \to H^2_T(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) \to H^2_T(\mathcal{O}_L, A^0_{p^\infty})^\Gamma \to 0.$$

Proof: This follows from the spectral sequence for relative equivariant cohomology (see the appendix to [10])

$$H^p(\Gamma, H^q_{T_\infty}(\mathcal{O}_L, A^0_{p^\infty})) \Rightarrow H^{p+q}_T(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}).$$

Note that $\mathrm{cd}_p(\Gamma) = 1$, and so this spectral sequence degenerates at the E_2 term. Furthermore, we know that $H^i_{T_\infty}(\mathcal{O}_L, A^0_{p^\infty}) = 0$ for $i \neq 1$ or 2 (see remark after Lemma 3.1). The spectral sequence then tells us that

$$\begin{aligned} H^1_T(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) &= H^0(\Gamma, H^1_{T_\infty}(\mathcal{O}_L, \Phi_{p^\infty})) \\ &= \bigoplus_{t \in T} H^0(\Gamma_t, {}_t\Phi(\mathcal{O}_{L_t})_{p^\infty}). \end{aligned}$$

It also gives us the required short exact sequence for $H^2_T(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})$, so it remains to show that $H^3_T(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) = 0$. But the spectral sequence shows that $H^3_T(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) = H^1(\Gamma, H^2_{T_\infty}(\mathcal{O}_L, A^0_{p^\infty}))$, so we will be done with the aid of the following lemma.

Lemma 3.4

$$H^1(\Gamma, H^2_{T_\infty}(\mathcal{O}_L, A^0_{p^\infty})) = 0$$

Proof:

$$\begin{aligned} H^1(\Gamma, H^2_{T_\infty}(\mathcal{O}_L, A^0_{p^\infty})) &= \varinjlim H^1(\Gamma/\Gamma_n, H^2_{T_n}(\mathcal{O}_{K_n}, A^0_{p^\infty})) \\ &= \varinjlim H^1(\Gamma/\Gamma_n, \bigoplus_{t \in T_\infty} H^2_\cdot(\mathcal{O}_{K_{n,t}}, A^0_{p^\infty})) \\ &= \varinjlim \bigoplus_{t \in T} H^1(\Gamma_t/\Gamma_{n,t}, H^2_\cdot(\mathcal{O}_{K_{n,t}}, A^0_{p^\infty})) \\ &= \bigoplus_{t \in T} \varinjlim H^1(\Gamma_t/\Gamma_{n,t}, H^2_\cdot(\mathcal{O}_{K_{n,t}}, A^0_{p^\infty})) \\ &= \bigoplus_{t \in T} \varinjlim H^1(\Gamma_t/\Gamma_{n,t}, H^2_\cdot(\mathcal{O}_{K_{n,t}}, A^0)_{p^\infty}) \end{aligned}$$

from the Kummer theory A^0, and the fact that $H^1_!(\mathcal{O}_{K_{n,t}}, A^0)$ is torsion. The relative cohomology sequence combined with $H^i(\mathcal{O}_{K_{n,t}}, A^0) = 0$ for $i = 1$ or 2 yields

$$\begin{aligned} H^1(\Gamma, H^2_{T_\infty}(\mathcal{O}_L, A^0_{p^\infty})) &= \bigoplus_{t\in T} \varinjlim H^1(\Gamma_t/\Gamma_{n,t}, H^1(K_{n,t}, A)_{p^\infty}) \\ &= \bigoplus_{t\in T} H^1(\Gamma_t, H^1(L_t, A)_{p^\infty}) \\ &= \bigoplus_{t\in T} H^1(\Gamma_t, H^1(L_t, A))_{p^\infty} \\ &= 0. \end{aligned}$$

The last zero comes from the fact that $H^1(\Gamma_t, H^1(L_t, A)) \hookrightarrow H^2(K_t, A)$ by the Hochshild-Serre spectral sequence, and $H^2(K_t, A) = 0$ by Tate's local duality. $\square$

We now consider maps from ordinary to equivariant cohomology. Let π be the map from $spec(\mathcal{O}_L) \to spec(\mathcal{O}_K)$. Then, there is a natural map (in fact, an injection) of sheaves

$$A^0_{p^\infty} \hookrightarrow \pi_\Gamma A^0_{p^\infty}.$$

This is essentially shown in the proof of lemma 3.7.1 in [10]. From it, we get natural maps

$$H^i(\mathcal{O}_K, A^0_{p^\infty}) \to H^i(\mathcal{O}_K, \pi_\Gamma A^0_{p^\infty}),$$

$$H^i_T(\mathcal{O}_K, A^0_{p^\infty}) \to H^i_T(\mathcal{O}_K, \pi_\Gamma A^0_{p^\infty})$$

and

$$H^i(\mathcal{O}_K - T, A^0_{p^\infty}) \to H^i(\mathcal{O}_K - T, \pi_\Gamma A^0_{p^\infty}).$$

Now, from the spectral sequences

$$H^p(\mathcal{O}_K, R^q\pi_\Gamma A^0_{p^\infty}) \Rightarrow H^{p+q}(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})$$

$$H^p(\mathcal{O}_K - T, R^q\pi_\Gamma A^0_{p^\infty}) \Rightarrow H^{p+q}(\mathcal{O}_L - T_\infty/\mathcal{O}_K - T, A^0_{p^\infty})$$

$$H^p_T(\mathcal{O}_K, R^q\pi_\Gamma A^0_{p^\infty}) \Rightarrow H^{p+q}_T(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})$$

(see appendix to [10]) we get maps

$$H^i(\mathcal{O}_K, \pi_\Gamma A^0_{p^\infty}) \to H^i(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})$$

$$H^i(\mathcal{O}_K - T, \pi_\Gamma A^0_{p^\infty}) \to H^i(\mathcal{O}_L - T_\infty/\mathcal{O}_K - T, A^0_{p^\infty}),$$

and

$$H^i_T(\mathcal{O}_K, \pi_\Gamma A^0_{p^\infty}) \to H^i_T(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}).$$

Composing, we get

$$\begin{array}{rcl} H^i_T(\mathcal{O}_K, A^0_{p^\infty}) & \to & H^i_T(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) \\ H^i(\mathcal{O}_K - T, A^0_{p^\infty}) & \to & H^i(\mathcal{O}_L - T_\infty/\mathcal{O}_K - T, A^0_{p^\infty}) \\ H^i(\mathcal{O}_K, A^0_{p^\infty}) & \to & H^i(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) \end{array} \tag{4}$$

Note that since the extension $\mathcal{O}_L - T_\infty/\mathcal{O}_K - T$ is unramified, the second map in 4 is an isomorphism.

We now examine $H^2_T(\mathcal{O}_K, A^0_{p^\infty}) \to H^2_T(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})$. Recall that $H^2_T(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})$ fits into the short exact sequence

$$0 \to H^1(\Gamma, H^1_{T_\infty}(\mathcal{O}_L, A^0_{p^\infty})) \to H^2_T(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) \to H^2_{T_\infty}(\mathcal{O}_L, A^0_{p^\infty})^\Gamma \to 0.$$

Above we showed that the map

$$H^2_T(\mathcal{O}_K, A^0_{p^\infty}) \to H^2_{T_\infty}(\mathcal{O}_L, A^0_{p^\infty})^\Gamma$$

has a finite kernel and that it is surjective. Combining this with the fact that it factors through $H^2_T(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})$ we get

Lemma 3.5 *The homomorphism*

$$H^2_T(\mathcal{O}_K, A^0_{p^\infty}) \to H^2_T(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})/(\text{Image } H^1_T(\mathcal{O}_L, A^0_{p^\infty})_\Gamma)$$

is surjective with finite kernel.

We can now deduce the analogous proposition

Proposition 3.6 *The homomorphism*

$$\theta: H^2(\mathcal{O}_K, A^0_{p^\infty}) \to H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})/H^1_T(\mathcal{O}_L/\mathcal{O}_K)$$

is surjective with finite kernel.

Proof: We just apply the following commutative diagram and the previous proposition.

$$\begin{array}{ccccccc} 0 & \to & H^1(\mathcal{O}_K - T, A^0_{p^\infty}) & \to & H^1(\mathcal{O}_L - T_\infty/\mathcal{O}_K - T, A^0_{p^\infty}) & \to & 0 \\ & & \downarrow & & \downarrow & & \\ & & H^2_T(\mathcal{O}_K, A^0_{p^\infty}) & \to & H^2_T(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) & & \\ & & \downarrow & & \downarrow & & \\ & & H^2(\mathcal{O}_K, A^0_{p^\infty}) & \to & H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) & & \\ & & \downarrow & & \downarrow & & \\ 0 & \to & H^2(\mathcal{O}_K - T, A^0_{p^\infty}) & \to & H^2(\mathcal{O}_L - T_\infty/\mathcal{O}_K - T, A^0_{p^\infty}) & \to & 0 \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

□

Finally, we show that α is a quasi-isomorphism.

Proposition 3.7 *The map $\alpha: \mathcal{S} \to \mathcal{S}_\infty^\Gamma$ is a quasi-isomorphism.*

Proof: The proof is essentially the same as the corresponding statement in [3] with the exception that we do not require good reduction at primes dividing p. Consider the following diagram.

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & A^0_{p^\infty}(\mathcal{O}_L)_\Gamma & & E_T \\
 & & & & \downarrow & & \downarrow \\
0 & \to & \mathcal{S} & \to & H^1(\mathcal{O}_K - T, A^0_{p^\infty}) & \to & H^2_T(\mathcal{O}_K, A^0_{p^\infty}) \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow \\
0 & \to & \mathcal{S}_\infty^\Gamma & \to & H^1(\mathcal{O}_L - T_\infty, A^0_{p^\infty})^\Gamma & \to & H^2_{T_\infty}(\mathcal{O}_L, A^0_{p^\infty})^\Gamma \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
\end{array}$$

The long horizontal rows are gotten by taking the relative cohomology sequences over $\mathcal{O}_K$ and $\mathcal{O}_L$. The vertical short exact sequence comes from the Hochshild-Serre spectral sequence applyed to the extension $\mathcal{O}_L - T_\infty / \mathcal{O}_K - T$. The bottom zeros come from the fact that $\mathrm{cd}_p(\Gamma) = 1$ and Lemma 3.2. A diagram chase then shows that the kernel of α is contained in $A^0_{p^\infty}(\mathcal{O}_L)_\Gamma$, and that its cokernel is a subquotient of E_T. Therefore, it suffices to show that these two groups are finite.

By [3] Lemma 6.7, $A^0_{p^\infty}(\mathcal{O}_L)_\Gamma$ is finite since $A^0_{p^\infty}(\mathcal{O}_L)^\Gamma$ is finite. For E_T, we apply Lemma 3.2. $\square$

4 Comparison of Height Pairings

In order to compare our height pairing with Schneider's analytic height, we we need to introduce some addition cohomological machinery. We introduce it here along with a few preliminary calculations to be used in the proof.

We will briefly need hypercohomology. If F is a sheaf in a category of abelian sheaves $\mathcal{A}$, then $F[n]$ will denote the complex

$$F \xrightarrow{n} F$$

in degree 0 and 1 in the derived category $D^+(\mathcal{A})$. This is the only type of complex of which we will take hypercohomology, so no confusion should ensue by not explicitly differentiating hypercohomology groups from cohomology groups. When dealing with $F[n]$, we always have a Kummer sequence in hypercohomology. That is, if H is a covariant functor on $\mathcal{A}$, then there is a long exact sequence ([11], p.356)

$$\cdots \to H^i(F[n]) \to H^i(F) \xrightarrow{n} H^i(F) \to H^{i+1}(F[n]) \to \cdots$$

Finally, if $F \xrightarrow{n} F$ is an epimorphism in $\mathcal{A}$, then $F[n]$ is canonically isomorphic to F_n, the kernel of multiplication by n on F. In this case, $H^i(F[n]) = H^i(F_n)$.

We will also use Schneider's modified cohomology. See [11] for its definition and basic properties. We include several of his constructions here for the reader's convenience. First we have the functors

$$Sh(\mathcal{O}_K - T) \xrightarrow{\mathcal{J}_*} cyl(\mathcal{O}_K) \xrightarrow{\mathcal{I}_t^!} \Gamma\text{-modules}.$$

Here, $cyl(\mathcal{O}_K)$ denotes the mapping cylinder of the functors $H^0(L_t, \alpha_t^* -)$ where α_t is the natural map

$$\alpha_t : spec(L_t) \to spec(\mathcal{O}_K - T)$$

induced by the inclusion of rings. The functors above are given by

$$\begin{array}{rcl} \mathcal{J}_* & : & \mathcal{F} \mapsto (\mathcal{F}; H^0(L_t, \alpha_t^* \mathcal{F}); \mathrm{id}) \\ \mathcal{I}_t^! & : & (\mathcal{F}; M_t; \mu_t) \mapsto \ker\, \mu_t. \end{array}$$

The i^{th} modified cohomology (and relative modified cohomology) group over a $spec(R)$ (relative to a place t) with coefficients in a sheaf $\mathcal{G} = (\mathcal{F}; M_t; \mu_t) \in cyl(\mathcal{O}_K)$ will be denoted $\hat{H}^i(R, \mathcal{G})$ (and $\hat{H}_t^i(R, \mathcal{G})$ respectively).

If G is a commutative group scheme over $\mathcal{O}_K - T$ which is locally of finite type, we let

$$NG(L_t) := \bigcup_{K_t \subset E \subset L_t} NG(E).$$

Recall that $NG(E)$ denotes the universal norms from L_t to E. Then, we have the functor to $cyl(\mathcal{O}_K)$,

$$NG := (G; NG(L_t); \text{inclusion}).$$

Finally, given a sheaf $\mathcal{F} \in Sh(\mathcal{O}_K)$, we define

$$M\mathcal{F} \in cyl(\mathcal{O}_K)$$

by

$$\mathcal{F} \mapsto (\mathcal{F}_{/\mathcal{O}_K - T}; H^0(\mathcal{O}_{L_t}, \alpha_t^* \mathcal{F}); \text{canonical}). \tag{5}$$

Note, that we will often write $\hat{H}^i(\mathcal{O}_K, \mathcal{F})$ for $\hat{H}^i(\mathcal{O}_K, M\mathcal{F})$.

Now, we calculate some local modified cohomology groups which we will need.

Proposition 4.1

$$\hat{H}_t^i(\mathcal{O}_K, \mathcal{J}_* A^0) = \begin{cases} H^1(L_t, A)^{\Gamma_t} & \textit{for} \quad i = 2 \\ 0 & \quad \textit{otherwise.} \end{cases}$$

$$\hat{H}^i_t(\mathcal{O}_K, A^0) = \begin{cases} 0 & \textit{for} \quad i = 0 \textit{ or } \geq 3 \\ H^0(\Gamma_t, \Phi(\mathcal{O}_{L_t})) & \textit{for} \qquad i = 1, \end{cases}$$

and there is a short exact sequence

$$0 \to H^1(\Gamma_t, \Phi(\mathcal{O}_{L_t})) \to \hat{H}^2_t(\mathcal{O}_K, A^0) \to H^1(L_t, A)^{\Gamma_t} \to 0$$

Proof: From

$$R^i\mathcal{I}^!_t(\mathcal{G}) = \begin{cases} \ker \mu_t & \text{for} \quad i = 0 \\ \operatorname{cok} \mu_t & \text{for} \quad i = 1 \\ H^{i-1}(L_t, \alpha^*_t\mathcal{F}) & \text{for} \quad i \geq 2 \end{cases} \tag{6}$$

(see [11], Proposition 1) we get

$$R^i\mathcal{I}^!_t A^0 = \begin{cases} 0 & \text{for} \quad i = 0 \\ \Phi(\mathcal{O}_L) & \text{for} \quad i = 1 \\ H^1(L_t, A) & \text{for} \quad i = 2 \\ 0 & \text{for} \quad i \geq 3 \end{cases}$$

and

$$R^i\mathcal{I}^!_t \mathcal{J}_* A^0 = \begin{cases} 0 & \text{for} \quad i = 0 \text{ or } 1 \\ H^1(L_t, A) & \text{for} \qquad i = 2 \\ 0 & \text{for} \qquad i \geq 3 \end{cases}$$

We then apply the spectral sequence

$$H^p(\Gamma, R^q\mathcal{I}^!_t(\mathcal{G})) \Rightarrow \hat{H}^{p+q}_t(\mathcal{O}_K, \mathcal{G}) \tag{7}$$

([11], Proposition 2) and use that $\mathrm{cd}_p(\Gamma) = 1$ and $H^1(L_t, A)_{\Gamma_t} = 0$ we get the proposition. □

Proposition 4.2 *There is a short exact sequence*

$$0 \to H^1_T(\mathcal{O}_L/\mathcal{O}_K) \to H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) \to \varinjlim \hat{H}^2(\mathcal{O}_K, \mathcal{J}_* A^0[p^\nu]) \to 0.$$

Proof: First, we note that

$$H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) \quad \text{can be identified with} \quad \varinjlim \hat{H}^2(\mathcal{O}_K, A^0[p^\nu])$$

by applying [11], Proposition 5 and Kummer sequences in cohomology and hypercohomology for A^0. From the relative cohomology sequences in modified cohomology, and the natural map $M \to \mathcal{J}_*$ we get

$$\begin{array}{ccc} \varinjlim H^1(\mathcal{O}_K - T, A^0[p^\nu]) & = & \varinjlim H^1(\mathcal{O}_K - T, A^0[p^\nu]) \\ \downarrow & & \downarrow \\ \bigoplus_{t \in T} \varinjlim \hat{H}^2_t(\mathcal{O}_K, A^0[p^\nu]) & \to & \bigoplus_{t \in T} \varinjlim \hat{H}^2_t(\mathcal{O}_K, \mathcal{J}_* A^0[p^\nu]) \\ \downarrow & & \downarrow \\ \varinjlim \hat{H}^2(\mathcal{O}_K, A^0[p^\nu]) & \to & \varinjlim \hat{H}^2(\mathcal{O}_K, \mathcal{J}_* A^0[p^\nu]) \\ \downarrow & & \downarrow \\ \varinjlim H^2(\mathcal{O}_K - T, A^0[p^\nu]) & = & \varinjlim H^2(\mathcal{O}_K - T, A^0[p^\nu]) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array} \tag{8}$$

The zeros at the bottom can be gotten as follows. The Kummer sequences in hypercohomology

$$0 \to \hat{H}_t^{i-1}(\mathcal{O}_K, A^0) \otimes \mathbf{Q}_p/\mathbf{Z}_p \to \varinjlim \hat{H}_t^i(\mathcal{O}_K, A^0[p^\nu]) \to \hat{H}_t^i(\mathcal{O}_K, A^0)_{p^\infty} \to 0$$

and

$$0 \to \hat{H}_t^{i-1}(\mathcal{O}_K, \mathcal{J}_* A^0) \otimes \mathbf{Q}_p/\mathbf{Z}_p \to \varinjlim \hat{H}_t^i(\mathcal{O}_K, \mathcal{J}_* A^0[p^\nu]) \to \hat{H}_t^i(\mathcal{O}_K, \mathcal{J}_* A^0)_{p^\infty} \to 0.$$

show that

$$\varinjlim \hat{H}_t^i(\mathcal{O}_K, A^0[p^\nu]) = \hat{H}_t^i(\mathcal{O}_K, A^0)_{p^\infty}$$

and

$$\varinjlim \hat{H}_t^i(\mathcal{O}_K, \mathcal{J}_* A^0[p^\nu]) = \hat{H}_t^i(\mathcal{O}_K, \mathcal{J}_* A^0)_{p^\infty}$$

for $i \geq 2$. We then apply Lemma 4.1 to get

$$\varinjlim \hat{H}_t^3(\mathcal{O}_K, A^0[p^\nu]) = \varinjlim \hat{H}_t^3(\mathcal{O}_K, \mathcal{J}_* A^0[p^\nu]) = 0$$

Similarly, we see that

$$\varinjlim \ker\, [\hat{H}_t^2(\mathcal{O}_K, A^0[p^\nu]) \to \hat{H}_t^2(\mathcal{O}_K, \mathcal{J}_* A^0[p^\nu])] = H^1(\Gamma, \Phi_{p^\infty})$$

Thus, a diagram chase in 8 shows that there is a short exact sequence

$$0 \to H^1(\Gamma, \Phi_{p^\infty}) \to H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) \to \varinjlim \hat{H}^2(\mathcal{O}_K, \mathcal{J}_* A^0[p^\nu]) \to 0$$

So, it only remains to show that the image of $H^1(\Gamma, \Phi_{p^\infty})$ in $H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})$ via $\varinjlim \hat{H}^2(\mathcal{O}_K, A^0[p^\nu])$ is the same as the image of $H^1_T(\mathcal{O}_L, A^0_{p^\infty})_\Gamma$. This can be shown by tracing from where the map from $H^1(\Gamma, \Phi_{p^\infty})$ came. □

4.1 Analytic Heights

In [9], Schneider defines an analytic p-adic height

$$(\ ,\ )^{Sch} \colon \tilde{A}(K) \times A(K) \to \mathbf{Q}_p.$$

In [11], he defines the following pairing

$$\begin{array}{ccccc}
\tilde{A}(K) & & & & \\
\| & & & & \\
\mathrm{Ext}^1_{\mathcal{O}_K}(A^0, \mathbf{G}_m) & & NA^0(\mathcal{O}_K) & & \\
\downarrow & & \| & & \\
\mathrm{Ext}^1_{cyl(\mathcal{O}_K)}(NA^0, N\mathbf{G}_m) & \times & \hat{H}^0(\mathcal{O}_K, NA^0) & \xrightarrow{\vee} & \hat{H}^1(\mathcal{O}_K, N\mathbf{G}_m) \\
& & & & \downarrow \mathrm{deg} \\
& & & & \mathbf{Z}_p
\end{array} \tag{9}$$

The product is the Yoneda product for modified cohomology. The definition of the degree map is given in [11], Section 5. The map

$$\mathrm{Ext}^1_{\mathcal{O}_K}(A^0, \mathbf{G}_m) \to \mathrm{Ext}^1_{cyl(\mathcal{O}_K)}(NA^0, N\mathbf{G}_m)$$

is defined by

$$(0 \to \mathbf{G}_m \to \mathcal{X} \to A^0 \to 0) \mapsto (0 \to N\mathbf{G}_m \to N\mathcal{X} \to NA^0 \to 0).$$

It exists by [9], Lemma 3. Finally, the identification

$$\tilde{A}(K) = \mathrm{Ext}^1_{\mathcal{O}_K}(A^0, \mathbf{G}_m)$$

is standard (say, using [4] (5.1)). So we get

$$(\ ,\ ) \colon \tilde{A}(K) \times NA^0(\mathcal{O}_K) \to \mathbf{Z}_p. \tag{10}$$

This can easily be seen as a pairing on points to $\mathbf{Q}_p$ since $NA^0(\mathcal{O}_K)$ is of finite index in $A(\mathcal{O}_K)$.

In [11], Section 7, Schneider proves that

$$(\ ,\ )^{Sch} = (\ ,\ ).$$

for abelian varieties with good ordinary reduction. His proof follows verbatim in our situation provided that the corestriction map

$$H^1(\Gamma_t, \tilde{A}(L_t)) \to H^1(\Gamma_{n,t}, \tilde{A}(L_t))$$

is surjective. We prove this here for ordinary semi-stable abelian varieties. Taking cohomology of Γ_t and $\Gamma_{n,t}$ for

$$0 \to \tilde{A}^0(\mathcal{O}_{L_t}) \to \tilde{A}(L_t) \to \tilde{\Phi}(\mathcal{O}_{L_t}) \to 0,$$

we get

$$\begin{array}{ccccccc}
H^1(\Gamma_{n,t}, \tilde{A}^0(\mathcal{O}_{L_t})) & \to & H^1(\Gamma_{n,t}, \tilde{A}(\mathcal{O}_{L_t})) & \to & H^1(\Gamma_{n,t}, \tilde{\Phi}) & \to & H^2(\Gamma_{n,t}, \tilde{A}^0(\mathcal{O}_{L_t})) \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} \\
H^1(\Gamma_t, \tilde{A}^0(\mathcal{O}_{L_t})) & \to & H^1(\Gamma_t, \tilde{A}(\mathcal{O}_{L_t})) & \to & H^1(\Gamma_t, \tilde{\Phi}) & \to & H^2(\Gamma_t, \tilde{A}^0(\mathcal{O}_{L_t}))
\end{array}$$

We need to show that a and c are surjective and d is injective. The calculations of Schneider ([10] pp. 283-284 and [11] pp. 359-360) apply here to $\tilde{A}^0$ even though it's twist matrix may have 1 as an eigenvalue. They show that a is surjective and that d is injective. Since Γ acts trivially on $\tilde{\Phi}$, one can easily check that c is an isomorphism (hence, surjective).

4.2 Algebraic vs. Analytic Height

We defined the pairing

$$\langle\langle\ ,\ \rangle\rangle_\gamma\colon H^1(\mathcal{O}_K, T_p(\tilde{A})) \times H^1(\mathcal{O}_K, T_p(A)) \to \mathbf{Q}_p$$

by diagram 2. Lemma 4 of [11] carries over in our situation to give the exact sequence

$$0 \to A^0(\mathcal{O}_K) \otimes \mathbf{Z}_p \to H^1(\mathcal{O}_K, T_p(A)) \to T_p(\text{III}_K(A)) \to 0.$$

Thus, by restriction, we get a pairing

$$\langle\ ,\ \rangle_\gamma\colon \tilde{A}^0(\mathcal{O}_K) \times A^0(\mathcal{O}_K) \to \mathbf{Q}_p$$

which can then be extended to $\tilde{A}(K) \times A(K)$. Note that if $\text{III}_K(A)$ (or even its p-primary component) is finite, which is conjecturally always the case, then $H^1(\mathcal{O}_K, T_p(A)) = A^0(\mathcal{O}_K)$. In this case, there is essentially no difference between $\langle\ ,\ \rangle_\gamma$ and $\langle\langle\ ,\ \rangle\rangle_\gamma$. The pairing $\langle\ ,\ \rangle_\gamma$ depends on our choice of generator γ of Γ, which we correct by setting

$$\langle\ ,\ \rangle_p := \langle\ ,\ \rangle_\gamma \log_p \kappa(\gamma).$$

We can now state and prove our main theorem.

Theorem 4.3

$$\langle\ ,\ \rangle_p = -(\ ,\ )^{Sch}$$

Proof: By the comments above, we need only show that $\langle\ ,\ \rangle_p = -(\ ,\ )$. We essentially follow Schneider's proof of the analogous proposition in [11]. Recall that $\langle\ ,\ \rangle_p$ is $\log_p \kappa(\gamma)$ times the pairing $\langle\ ,\ \rangle_\gamma$ which is induced by diagram 2.

Taking duals, we can equivalently define it by

$$\begin{array}{ccccc}
\tilde{A}^0(\mathcal{O}_K) \otimes \mathbf{Z}_p & & A^0(\mathcal{O}_K) \otimes \mathbf{Q}_p/\mathbf{Z}_p & & \\
\Big| & & \downarrow & & \\
\Big| & & H^1(\mathcal{O}_K, A^0_{p^\infty}) & & \\
\Big| & & \downarrow & & \\
\Big| & & \mathcal{S} & & \\
\Big| & & \downarrow & & \\
\Big| & & \mathcal{S}_\infty^\Gamma & & \\
\Big| & & \downarrow & & \\
\Big| & & (\mathcal{S}_\infty)_\Gamma & & \\
\Big| & & \uparrow & & \\
\Big| & & H^1(\mathcal{O}_L, A^0_{p^\infty})_\Gamma / H^1_T(\mathcal{O}_L) & & \\
\Big| & & \downarrow & & \\
\Big| & & H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty}) / H^1_T(\mathcal{O}_L/\mathcal{O}_K) & & \\
\Big| & & \uparrow & & \\
\Big| & & H^2(\mathcal{O}_K, A^0_{p^\infty}) & & \\
\downarrow & & \downarrow & & \\
\varprojlim H^1(\mathcal{O}_K, \tilde{A}_{p^\nu}) & \times & H^2(\mathcal{O}_K, A_{p^\infty}) & \to & H^3(\mathcal{O}_K, \mathbf{G}_m) \\
 & & & & \downarrow \\
 & & & & \mathbf{Q}/\mathbf{Z}
\end{array}$$

using the identification

$$\mathrm{Hom}(A^0(\mathcal{O}_K) \otimes \mathbf{Q}_p/\mathbf{Z}_p, \mathbf{Q}_p/\mathbf{Z}_p) = \mathrm{Hom}_{\mathbf{Z}_p}(A^0(\mathcal{O}_K) \otimes \mathbf{Z}_p, \mathbf{Z}_p).$$

We begin by transforming the right column.

Lemma 4.4 *The following diagram is commutative:*

$$\begin{array}{ccc}
H^1(\mathcal{O}_K, A^0_{p^\infty}) & \to & \mathcal{S} \\
\downarrow & & \downarrow \\
H^1(\mathcal{O}_L, A^0_{p^\infty})^\Gamma & \to & \mathcal{S}_\infty^\Gamma \\
\downarrow & & \Big| \\
H^1(\mathcal{O}_L, A^0_{p^\infty})_\Gamma & & \downarrow \\
\downarrow & & \\
H^1(\mathcal{O}_L, A^0_{p^\infty})_\Gamma / H^1_T(\mathcal{O}_L) & \to & (\mathcal{S}_\infty)_\Gamma
\end{array}$$

Proof: This follows from the definitions of $\mathcal{S}$ and $\mathcal{S}_\infty$. □

This will remove the classical Selmer group from the right column. It can now be written in terms of modified cohomology with the next lemma.

Lemma 4.5 *The following diagram is commutative.*

$$\begin{array}{ccccccc}
NA^0(\mathcal{O}_K)/p^\nu & \to & \hat{H}^0(\mathcal{O}_K, \mathcal{J}_* A^0[p^\nu]) & \leftarrow & \hat{H}^0(\mathcal{O}_K, A^0)/p^\nu & = \\
\downarrow & & \downarrow & & \downarrow & \\
\hat{H}^1(\mathcal{O}_K, NA^0[p^\nu]) & \to & \hat{H}^1(\mathcal{O}_K, \mathcal{J}_* A^0[p^\nu]) & \leftarrow & \hat{H}^1(\mathcal{O}_K, A^0[p^\nu]) & \leftarrow \\
\downarrow & & \downarrow & & \downarrow & \\
\hat{H}^1(\mathcal{O}_L, NA^0[p^\nu])^\Gamma & \to & \hat{H}^1(\mathcal{O}_L, \mathcal{J}_* A^0[p^\nu])^\Gamma & \leftarrow & \hat{H}^1(\mathcal{O}_L, A^0[p^\nu])^\Gamma & = \\
\downarrow & & \downarrow & & \downarrow & \\
\hat{H}^1(\mathcal{O}_L, NA^0[p^\nu])_\Gamma & \to & \hat{H}^1(\mathcal{O}_L, \mathcal{J}_* A^0[p^\nu])_\Gamma & \leftarrow & \hat{H}^1(\mathcal{O}_L, A^0[p^\nu])_\Gamma & = \\
\downarrow & & \downarrow & & \downarrow & \\
\hat{H}^2(\mathcal{O}_K, NA^0[p^\nu]) & \to & \hat{H}^2(\mathcal{O}_K, \mathcal{J}_* A^0[p^\nu]) & \leftarrow & \hat{H}^2(\mathcal{O}_K, A^0[p^\nu]) & =
\end{array}$$

$$\begin{array}{ccccc}
= & A^0(\mathcal{O}_K)/p^\nu & = & A^0(\mathcal{O}_K)/p^\nu \\
 & \downarrow & & \downarrow \\
\leftarrow & H^1(\mathcal{O}_K, A^0[p^\nu]) & = & H^1(\mathcal{O}_K, A^0_{p^\nu}) \\
 & \downarrow & & \downarrow \\
= & H^1(\mathcal{O}_L, A^0[p^\nu])^\Gamma & = & H^1(\mathcal{O}_L, A^0_{p^\nu})^\Gamma \\
 & \downarrow & & \downarrow \\
= & H^1(\mathcal{O}_L, A^0[p^\nu])_\Gamma & = & H^1(\mathcal{O}_L, A^0_{p^\nu})_\Gamma \\
 & \downarrow & & \downarrow \\
= & H^2(\mathcal{O}_L/\mathcal{O}_K, A^0[p^\nu]) & = & H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\nu}) \\
 & & & \downarrow \\
 & & & H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\nu})/H^1_T(\mathcal{O}_L/\mathcal{O}_K)
\end{array}$$

Proof: The maps among the first three columns come from the morphisms of functors

$$N \to \mathcal{J}_* \leftarrow M$$

where the functor M is implicit in the third column.

The maps between the third column and the first of the second group come from Proposition 5 of [11] which, for a connected smooth group scheme G, identifies

$$\hat{H}^i(\mathcal{O}_K, G) \quad \text{with} \quad H^i(\mathcal{O}_L/\mathcal{O}_K, G)$$

and

$$\hat{H}^i(\mathcal{O}_L, G) \quad \text{with} \quad H^i(\mathcal{O}_L, G).$$

This extends to $A^0[p^\nu]$ by applying it to A^0 and using the Kummer sequence in hypercohomology.

The equalities between the two columns of the second group come from the identification of the hypercohomology of the complex $A^0[p^\nu]$ with the cohomology of the group scheme $A^0_{p^\nu}$, because we are working on the flat site where $A^0 \xrightarrow{p^\nu} A^0$ is surjective. □

Note, that the maps in between the second and third columns go the wrong way for making the transition *a priori* all the way to the first column for the height pairing. However, Proposition 4.2 shows that in the direct limit, the map

$$H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\nu}) \to \hat{H}^2(\mathcal{O}_K, \mathcal{J}_*A^0[p^\nu])$$

factors through $H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})/H^1_T(\mathcal{O}_L/\mathcal{O}_K)$, and that

$$H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})/H^1_T(\mathcal{O}_L/\mathcal{O}_K) \to \varinjlim \hat{H}^2(\mathcal{O}_K, \mathcal{J}_*A^0[p^\nu])$$

is a quasi-isomorphism. Composing, we get that the map

$$H^2(\mathcal{O}_K, A^0_{p^\infty}) \to \varinjlim \hat{H}^2(\mathcal{O}_K, \mathcal{J}_*A^0[p^\nu])$$

is also a quasi-isomorphism. So we may rewrite the pairing as

$$\begin{tikzcd}
\tilde{A}^0(\mathcal{O}_K)\otimes \mathbf{Z}_p \arrow[ddddddd] & & NA^0(\mathcal{O}_K)\otimes \mathbf{Q}_p/\mathbf{Z}_p \arrow[d] & & \\
& & \varinjlim \hat{H}^1(\mathcal{O}_K, NA^0[p^\nu]) \arrow[d] & & \\
& & \varinjlim \hat{H}^1(\mathcal{O}_L, NA^0[p^\nu])^\Gamma \arrow[d] & & \\
& & \varinjlim \hat{H}^1(\mathcal{O}_L, NA^0[p^\nu])_\Gamma \arrow[d] & & \\
& & \varinjlim \hat{H}^2(\mathcal{O}_K, NA^0[p^\nu]) \arrow[d] & & \\
& & \varinjlim \hat{H}^2(\mathcal{O}_K, \mathcal{J}_*A^0[p^\nu]) & & \\
& & H^2(\mathcal{O}_K, A^0_{p^\infty}) \arrow[u] \arrow[d] & & \\
\varprojlim H^1(\mathcal{O}_K, \tilde{A}_{p^\nu}) & \times & H^2(\mathcal{O}_K, A_{p^\infty}) \arrow[r] & H^3(\mathcal{O}_K, \mathbf{G}_m) \arrow[d] \\
& & & \mathbf{Q}/\mathbf{Z}
\end{tikzcd}$$

We rewrite the pairing at the bottom of the diagram as an Ext-pairing. The map ϵ_1 is an edge map from the local-global Ext-spectral sequence.

$$\begin{tikzcd}
H^1(\mathcal{O}_K, \tilde{A}_{p^\nu}) \arrow[d] & \times & H^2(\mathcal{O}_K, A_{p^\nu}) \arrow[dd, equal] & \rightarrow & H^3(\mathcal{O}_K, \boldsymbol{\mu}_{p^\nu}) \arrow[dd, equal] & = & \mathbf{Z}/p^\nu\mathbf{Z} \\
H^1(\mathcal{O}_K, \underline{\mathrm{Hom}}_{\mathcal{O}_K}(A_{p^\nu}, \boldsymbol{\mu}_{p^\nu})) \arrow[d, "\epsilon_1"] & & & & & & \\
\mathrm{Ext}^1_{\mathcal{O}_K}(A_{p^\nu}, \boldsymbol{\mu}_{p^\nu}) & \times & H^2(\mathcal{O}_K, A_{p^\nu}) & \rightarrow & H^3(\mathcal{O}_K, \boldsymbol{\mu}_{p^\nu}) & = & \mathbf{Z}/p^\nu\mathbf{Z}
\end{tikzcd}$$

By functoriality, we furthermore have the commutative diagrams

$$\begin{array}{ccccccc}
\mathrm{Ext}^1_{\mathcal{O}_K}(A^0_{p^\nu}, \boldsymbol{\mu}_{p^\nu}) & \times & H^2(\mathcal{O}_K, A^0_{p^\nu}) & \to & H^3(\mathcal{O}_K, \boldsymbol{\mu}_{p^\nu}) & = & \mathbf{Z}/p^\nu\mathbf{Z} \\
\uparrow & & \downarrow & & \| & & \\
\mathrm{Ext}^1_{\mathcal{O}_K}(A_{p^\nu}, \boldsymbol{\mu}_{p^\nu}) & \times & H^2(\mathcal{O}_K, A_{p^\nu}) & \to & H^3(\mathcal{O}_K, \boldsymbol{\mu}_{p^\nu}) & = & \mathbf{Z}/p^\nu\mathbf{Z}
\end{array}$$

and

$$\begin{array}{ccc}
H^1(\mathcal{O}_K, \tilde{A}^0_{p^\nu}) & \longrightarrow & H^1(\mathcal{O}_K, A_{p^\nu}) \\
\downarrow & & \downarrow \\
H^1(\mathcal{O}_K, \underline{\mathrm{Hom}}_{\mathcal{O}_K}(A^0_{p^\nu}, \boldsymbol{\mu}_{p^\nu})) & & H^1(\mathcal{O}_K, \underline{\mathrm{Hom}}_{\mathcal{O}_K}(A_{p^\nu}, \boldsymbol{\mu}_{p^\nu})) \\
\downarrow{\scriptstyle \epsilon_1} & & \downarrow{\scriptstyle \epsilon_1} \\
\mathrm{Ext}^1_{\mathcal{O}_K}(A^0_{p^\nu}, \boldsymbol{\mu}_{p^\nu}) & \longleftarrow & \mathrm{Ext}^1_{p^\nu}(A_{p^\nu}, \boldsymbol{\mu}_{p^\nu})
\end{array}$$

Using these diagrams, we can transform the diagram for $(\log_p \kappa(\gamma))^{-1}\langle\ ,\ \rangle_p$ to be

$$\begin{array}{ccccc}
\tilde{A}^0(\mathcal{O}_K) \otimes \mathbf{Z}_p & & NA^0(\mathcal{O}_K) \otimes \mathbf{Q}_p/\mathbf{Z}_p & & \\
\downarrow & & \downarrow & & \\
\varprojlim H^1(\mathcal{O}_K, \tilde{A}^0_{p^\nu}) & & \varinjlim \hat{H}^1(\mathcal{O}_K, NA^0[p^\nu]) & & \\
\Big\downarrow & & \downarrow & & \\
 & & \varinjlim \hat{H}^1(\mathcal{O}_L, NA^0[p^\nu])^\Gamma & & \\
 & & \downarrow & & \\
 & & \varinjlim \hat{H}^1(\mathcal{O}_L, NA^0[p^\nu])_\Gamma & & \\
 & & \downarrow & & \\
 & & \varinjlim \hat{H}^2(\mathcal{O}_K, NA^0[p^\nu]) & & \\
\varprojlim H^1(\mathcal{O}_K, \underline{\mathrm{Hom}}_{\mathcal{O}_K}(A^0_{p^\nu}, \boldsymbol{\mu}_{p^\nu})) & & \downarrow & & \\
\downarrow{\scriptstyle \epsilon_1} & & \varinjlim \hat{H}^2(\mathcal{O}_K, \mathcal{J}_* A^0[p^\nu]) & & \\
 & & \uparrow & & \\
\varprojlim \mathrm{Ext}^1_{\mathcal{O}_K}(A^0[p^\nu], \boldsymbol{\mu}_{p^\nu}) & \times & H^2(\mathcal{O}_K, A^0_{p^\infty}) & \longrightarrow & H^3(\mathcal{O}_K, \mathbf{G}_m) \\
 & & & & \downarrow \\
 & & & & \mathbf{Q}/\mathbf{Z}
\end{array}$$

We can further transform the left column using the following commutative diagram of Schneider ([11], p.63).

$$\begin{array}{ccc}
H^1(\mathcal{O}_K, \tilde{A}^0_{p^\nu}) & \xleftarrow{-\delta} & H^0(\mathcal{O}_K, \tilde{A}^0) \\
\downarrow & & \downarrow \\
H^1(\mathcal{O}_K, \underline{\mathrm{Hom}}_{\mathcal{O}_K}(A^0[p^\nu], \boldsymbol{\mu}_{p^\nu})) & & H^0(\mathcal{O}_K, \underline{\mathrm{Ext}}^1_{\mathcal{O}_K}(A^0, \mathbf{G}_m)) \\
\downarrow{\scriptstyle \epsilon_1} & & \uparrow{\scriptstyle \epsilon_2} \\
\mathrm{Ext}^1_{\mathcal{O}_K}(A^0[p^\nu], \boldsymbol{\mu}_{p^\nu}) & \xleftarrow{r} & \mathrm{Ext}^1_{\mathcal{O}_K}(A^0, \mathbf{G}_m)
\end{array}$$

The map ϵ_2 is also an edge morphism from the local-global Ext-spectral sequence. It is, in fact, an isomorphism. The map δ is the connecting homomorphism in the long exact sequence in cohomology. Although Schneider works in the case of good ordinary reduction, the proof of the commutativity of the diagram is mostly formal, and carries over for semi-stable A. Using it, we can alter the pairing to be given by the diagram

$$
\begin{array}{ccccc}
\tilde{A}^0(\mathcal{O}_K)\otimes \mathbf{Z}_p & & NA^0(\mathcal{O}_K)\otimes \mathbf{Q}_p/\mathbf{Z}_p & & \\
\downarrow & & \downarrow & & \\
\operatorname{Ext}^1_{\mathcal{O}_K}(A^0,\mathbf{G}_m)\otimes \mathbf{Z}_p & & \varinjlim \hat{H}^1(\mathcal{O}_K, NA^0[p^\nu]) & & \\
\Big\downarrow & & \downarrow & & \\
\Big\downarrow & & \varinjlim \hat{H}^1(\mathcal{O}_L, NA^0[p^\nu])^\Gamma & & \\
\Big\downarrow & & \downarrow & & \\
\Big\downarrow & & \varinjlim \hat{H}^1(\mathcal{O}_L, NA^0[p^\nu])_\Gamma & & \\
\Big\downarrow {\scriptstyle -r} & & \downarrow & & \\
\Big\downarrow & & \varinjlim \hat{H}^2(\mathcal{O}_K, NA^0[p^\nu]) & & \\
\Big\downarrow & & \downarrow & & \\
\Big\downarrow & & \varinjlim \hat{H}^2(\mathcal{O}_K, \mathcal{J}_* A^0[p^\nu]) & & \\
\Big\downarrow & & \uparrow & & \\
\varprojlim \operatorname{Ext}^1_{\mathcal{O}_K}(A^0[p^\nu],\boldsymbol{\mu}_{p^\nu}) & \times & H^2(\mathcal{O}_K, A^0_{p^\infty}) & \to & H^3(\mathcal{O}_K,\mathbf{G}_m) \\
 & & & & \downarrow \\
 & & & & \mathbf{Q}/\mathbf{Z}
\end{array}
$$

The final major step is to change the pairing to one of modified cohomology groups. We denote by $\hat{r}$ the composite map

$$\operatorname{Ext}^1_{\mathcal{O}_K}(A^0,\mathbf{G}_m)\to \operatorname{Ext}^1_{cyl(\mathcal{O}_K)}(NA^0,N\mathbf{G}_m)\to \varprojlim \operatorname{Ext}^1_{cyl(\mathcal{O}_K)}(NA^0[p^\nu],N\mathbf{G}_m[p^\nu]).$$

The second map comes from the functoriality of the $[p^\nu]$-construction.

Note that the lower right portion of the above diagram could be replaced simply with the natural map

$$H^2(\mathcal{O}_K, A^0_{p^\infty})\to H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})\to \varinjlim \hat{H}^2(\mathcal{O}_K,\mathcal{J}_*A^0[p^\nu]).$$

With this in mind we state the following lemma. We will use the following shorthand for the diagram.

$$E^1_{cyl(\mathcal{O}_K)} \quad \text{will denote} \quad \operatorname{Ext}^1_{cyl(\mathcal{O}_K)}(NA^0[p^\nu],N\mathbf{G}_m[p^\nu])$$

and all of the cohomology groups will be over $spec(\mathcal{O}_K)$, so we will omit it from the notation.

Lemma 4.6 *In the diagram*

$$
\begin{array}{ccccc}
\varprojlim E^1_{cyl(\mathcal{O}_K)} & \times & \varinjlim \hat{H}^2(NA^0[p^\nu]) & \to & \varinjlim \hat{H}^3(N\mathbf{G}_m[p^\nu]) \\
\uparrow{\scriptstyle \hat{r}} & & \downarrow & & \downarrow{\scriptstyle d} \\
\mathrm{Ext}^1_{\mathcal{O}_K}(A^0, \mathbf{G}_m) & & \varinjlim \hat{H}^2(\mathcal{J}_* A^0[p^\nu]) & & \mathbf{Q}_p/\mathbf{Z}_p \\
\downarrow{\scriptstyle r} & & \uparrow & & \| \\
\varprojlim \mathrm{Ext}^1_{\mathcal{O}_K}(A^0_{p^\nu}, \boldsymbol{\mu}_{p^\nu}) & \times & H^2(A^0_{p^\infty}) & \to & H^3(\boldsymbol{\mu}_{p^\infty})
\end{array}
$$

there exists a positive number C such that, for any $e \in \mathrm{Ext}^1_{\mathcal{O}_K}(A^0, \mathbf{G}_m)$, and for any

$$x \in H^2(\mathcal{O}_K, A^0_{p^\infty}), \quad \textit{and} \quad y \in \hat{H}^2(\mathcal{O}_K, NA^0[p^\nu])$$

such that x and y map to the same element of $\varinjlim \hat{H}^2(\mathcal{O}_K, \mathcal{J}_* A^0[p^\nu])$, we have that

$$C(r(e) \times x - d(\hat{r}(e) \times y)) = 0.$$

Proof: This lemma, and it's proof are essentially the same as Lemma 6.4 of [11]. We have substituted $\varinjlim \hat{H}^2(\mathcal{J}_* A^0[p^\nu])$ for $H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})$. This comes up in the proof where we must show that

$$\varinjlim \hat{H}^2(\mathcal{O}_K, NA^0[p^\nu]) \to \varinjlim \hat{H}^2(\mathcal{O}_K, \mathcal{J}_* A^0[p^\nu]),$$

has a finite kernel. But, it is bounded by the order of $\hat{H}^1(\mathcal{O}_K, P)$, where $P \in cyl(\mathcal{O}_K)$ is the skyscraper sheaf $(0; A^0_{p^\infty}(L_v)/NA^0_{p^\infty}(L_v); 0)$. For skyscraper sheaves of the form $Y = (0; X; 0)$, $\hat{H}^i(\mathcal{O}_K, Y) = H^i(\Gamma, X)$. In our case, we have the short exact sequence

$$0 \to NA^0_{p^\infty}(L_v) \to A^0_{p^\infty}(L_v) \to A^0_{p^\infty}(L_v)/NA^0_{p^\infty}(L_v) \to 0$$

so it suffices to show that $H^0(\Gamma, A^0_{p^\infty}(L_v))$ and $H^1(\Gamma, NA^0_{p^\infty}(L_v))$ are finite. But $H^0(\Gamma, A^0_{p^\infty}(L_v)) = A^0_{p^\infty}(K_v)$ which is obviously finite. This in turn, implies that $H^0(\Gamma, NA^0_{p^\infty}(L_v))$ is finite, which shows that $H^1(\Gamma, NA^0_{p^\infty}(L_v))$ is finite by [3], Lemma 6.7. $\square$

Since $NA^0(\mathcal{O}_K) \otimes \mathbf{Q}_p/\mathbf{Z}_p$ is a divisible group, the diagram in Lemma 4.6 must in fact commute for the elements which come from this group. Therefore, we can rewrite the pairing as

$$
\begin{array}{ccc}
\tilde{A}^0(\mathcal{O}_K)\otimes\mathbf{Z}_p & & NA^0(\mathcal{O}_K)\otimes\mathbf{Q}_p/\mathbf{Z}_p \\
 & & \downarrow \\
\Big\downarrow & & \varinjlim \hat{H}^1(\mathcal{O}_K, NA^0[p^\nu]) \\
 & & \downarrow \\
\mathrm{Ext}^1_{\mathcal{O}_K}(A^0, \mathbf{G}_m)\otimes\mathbf{Z}_p & & \varinjlim \hat{H}^1(\mathcal{O}_L, NA^0[p^\nu])^\Gamma \\
 & & \downarrow \\
\Big\downarrow{\scriptstyle -\hat{r}} & & \varinjlim \hat{H}^1(\mathcal{O}_L, NA^0[p^\nu])_\Gamma \\
 & & \downarrow \\
\varprojlim \mathrm{Ext}^1_{cyl(\mathcal{O}_K)}(NA^0[p^\nu], N\mathbf{G}_m[p^\nu]) & \times & \varinjlim \hat{H}^2(\mathcal{O}_K, NA^0[p^\nu]) \\
 & & \Big\downarrow{\scriptstyle \vee} \\
 & & \varinjlim \hat{H}^3(\mathcal{O}_K, N\mathbf{G}_m[p^\nu]) \\
 & & \Big\downarrow{\scriptstyle d} \\
 & & \mathbf{Q}_p/\mathbf{Z}_p
\end{array}
$$

Schneider's argument (cf, [11], p. 369) then carries over to finish the proof of Theorem 4.3. □

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