ZERO FREE REGIONS FOR A RATIONAL FUNCTION WITH APPLICATIONS

JOHN W. JONES [1] **and BRUNO D. WELFERT**

Department of Mathematics, Arizona State University, Tempe, Arizona 85287-1804.

In this paper, we study a rational function which plays an important role in several problems of interest (eigenvalue problems, linear control theory,...). Our main interest is to determine zero-free regions. We also derive upper and lower bounds for this function.

1. Introduction

For $z \in \mathbb{C}$, we consider the rational function

$$\varphi_n(z) \stackrel{def}{=} \sum_{k=0}^{2n} \frac{(-1)^k}{z - p_k}, \tag{1}$$

where $n > 0$ and the p_k are real numbers satisfying the interlacing property:

$$p_k < p_{k+1} \qquad \text{for } 0 \le k < 2n. \tag{2}$$

Our goal is to locate zero-free regions of φ_n and to determine a lower bound on the imaginary part of the roots of the zeros of φ_n. In Section 2, we relate this question to several problems ranging from eigenpair determination to controllability of linear systems. In Section 3 we derive lower and upper bounds for $\varphi_n(z)$ on the real axis, and show in particular that its zeros are complex, and that the contribution of each terms $\dfrac{(-1)^k}{z - p_k}$ in (1) is essentially local. In Section 4 we identify zero free regions of the complex plane, and show that the roots z of φ_n are bounded away from the real axis, namely that

$$|\Im m(z)| \ge \frac{\sqrt{3}}{2} d,$$

where d is the minimal distance between successive p_k, i.e., $d = \min\limits_{0 \le k < 2n} (p_{k+1} - p_k)$.

An analysis of the limiting case $p_{k+1} \longrightarrow p_k$ is presented in Section 5. We show that φ_n can then have roots arbitrarily close to the real axis[2].

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[2] It is understood here that p_0 and p_{2n} are *fixed*. The distance $p_{2n} - p_0$ represents a scale of the problem. By making this scale small enough, one could eventually obtain zeros as close to the real axis as desired.

2. The function φ_n and related problems

For $n \geq 1$ and $z \in \mathbb{C} \setminus \{p_0, \ldots, p_{2n}\}$ we introduce the polynomial functions

$$P_n(z) = \prod_{k=0}^{n} (z - p_{2k}), \tag{3}$$

$$Q_n(z) = \prod_{k=1}^{n} (z - p_{2k-1}), \tag{4}$$

$$\phi_n(z) = P_n(z) Q_n(z) \varphi_n(z), \tag{5}$$

as well as the rational function

$$f_n(z) = \frac{Q_n(z)}{P_n(z)}.$$

The interlacing property (2) implies that P_n and Q_n are coprime. Given P_n, an example of a Q_n satisfying (2) is P_n'. Remark that $f_n'(z) = -f_n(z)\varphi_n(z)$.

In this Section we present a few problems and results involving directly either φ_n or ϕ_n.

2.1. Inverse eigenvalue problem. Consider the problem of finding linear constraints $c^t x = 0$ that yield preassigned stationary values $p_{2k-1}, 1 \leq k \leq n$ to the quadratic form $x^t A x$, where A is symmetric with spectral decomposition $Q\ diag(p_0, p_2, \ldots, p_{2n})\ Q^t$ (inverse eigenvalue problem). The solution to this problem is (see [3])

$$c = Q d \quad \text{with} \quad d_k^2 = \frac{1}{(1/f_n)'(p_{2k})} \quad \text{for } 0 \leq k \leq n,$$

i.e.,

$$d_k^2 = -\lim_{z \longrightarrow p_{2k}} \frac{f_n^2(z)}{f_n'(z)} = \lim_{z \longrightarrow p_{2k}} \frac{f_n(z)}{\varphi_n(z)}.$$

Thus, the computation of d (and c) relies on the accurate determination of the coefficients d_k. In particular, we are interested in determining the location of the zeros of φ_n.

2.2. Linear control theory. For $n \geq 1$ and $z_1, z_2 \in \mathbb{C} \setminus \{p_0, \ldots, p_{2n}\}$ we introduce the two-variable polynomial function $\tilde{\phi}_n$ defined by

$$\tilde{\phi}_n(z_1, z_2) = \frac{P_n(z_1)Q_n(z_2) - P_n(z_2)Q_n(z_1)}{z_1 - z_2}.$$

It is easy to check that

$$\tilde{\phi}_n(z, z) = P_n'(z)Q_n(z) - P_n(z)Q_n'(z) = \phi_n(z).$$

On the other hand,

$$\tilde{\phi}_n(z_1, z_2) = \sum_{j,m=0}^{n-1} b_{j,m} z_1^j z_2^m = Z_1^t B_n Z_2,$$

for some $n \times n$ matrix B_n, where

$$Z_i = \left(1, z_i, \ldots, z_i^{n-1}\right)^t \quad i = 1, 2.$$

The function $\tilde{\phi}_n$ is known in linear control theory as the generating function of the symmetric matrix B_n. The latter is also referred to as the Bezoutian matrix. The non-singularity of B_n (and therefore the sign invariance of ϕ_n on $\mathbb{R}$) can be shown to be closely related to the complete controllability and observability of a realization of the rational function f_n [1]. Its condition depends in particular on how close a root of ϕ_n is from the real axis.

The Bezoutian matrix B_n has also been characterized as the solution of certain Lyapunov equations [4].

Note that because of the condition (2), the polynomials P_n and Q_n can be considered as consecutive members of a sequence of polynomials orthogonal for a positive definite moment functional [2]. This implies that there exists positive values $\mu_i, 0 \leq i \leq n-1$ such that

$$\tilde{\phi}_n(z_1, z_2) = \sum_{i=0}^{n-1} \mu_i \tilde{Q}_i(z_1)\tilde{Q}_i(z_2),$$

where the degree of $\tilde{Q}_i = i$ $(0 \leq i \leq n)$ and $\tilde{Q}_n = Q_n$ $(\mu_n = 1)$, and thus $\phi_n(z) > 0$ for $z \in \mathbb{R}$.

2.3. Rational Krylov methods. Given a $p \times p$ symmetric matrix A, with normalized eigenpairs (λ_j, y_j), and $2n+1$ reals $p_0, p_2, \ldots, p_{2n}$ satisfying the interlacing property 2 such that $p_{2k} \neq \lambda_j$ for $0 \leq k \leq n$, we define a sequence of vectors $\{x^{(i)}\}_{i \geq 0}$ by [5]:

$$\begin{cases} x^{(0)} & = & \displaystyle\sum_{j=1}^{p} \alpha_j y_j \quad \text{given}, \\ x^{(k+1)} & = & (A - p_{2k} I)^{-1} x^{(k)} \text{ for } \; k \geq 0. \end{cases}$$

Consider the polynomials P_n and Q_n defined by (3) and (4). A Newton form of Q_n using the centers $p_2, p_4, \ldots, p_{2n}$ is

$$Q_n(\lambda) = \sum_{i=1}^{n+1} \left(\beta_i \prod_{j=i}^{n} (\lambda - p_{2j}) \right)$$

with $\beta_1 = 1$.

Then the residual

$$r^{(n)} \stackrel{def}{=} \sum_{i=1}^{n+1} \beta_i x^{(i)}$$

can be shown to be equal to

$$r^{(n)} = \sum_{j=1}^{p} \alpha_j f_n(\lambda_j) y_j \,.$$

As n increases, the sequence $\{x^{(i)}\}_{1 \leq i \leq n+1}$ becomes linearly dependent. If $p_{2j-1} \simeq \lambda_k$ for some $1 \leq j < n$ and $1 \leq k \leq p$, then $f_n(\lambda) \simeq (\lambda - \lambda_k) g_n(\lambda)$, and the vector

$x \stackrel{def}{=} \dfrac{g_n(A)x^{(0)}}{\|g_n(A)x^{(0)}\|_2}$ satisfies

$$Ax - \lambda_k x \simeq \frac{1}{\|g_n(A)x^{(0)}\|_2} r^{(n)}.$$

Since

$$\begin{aligned} \|g_n(A)x^{(0)}\|_2^2 &= \sum_{j=1}^{p} \alpha_j^2 g_n^2(\lambda_j) = \sum_{j=1}^{p} \alpha_j^2 \left(f_n'(\lambda_j)\right)^2 = \sum_{j=1}^{p} \alpha_j^2 f_n^2(\lambda_j)\varphi_n^2(\lambda_j) \\ &\geq \min_{j,\alpha_j \neq 0} \varphi_n^2(\lambda_j) \sum_{j=1}^{p} \alpha_j^2 f_n^2(\lambda_j) = \min_{j,\alpha_j \neq 0} \varphi_n^2(\lambda_j)\, ||r^{(n)}||_2^2, \end{aligned}$$

we have

$$||Ax - \lambda_k x|| \lesssim \frac{1}{\min_{j,\alpha_j \neq 0} |\varphi_n(\lambda_j)|}$$

i.e., x is an approximate eigenvector if $|\varphi_n(\lambda_j)|$ is large enough. This depends essentially on the location of the zeros of φ_n in the complex plane, in particular how far these roots are from the real axis.

2.4. Reduction to tridiagonal/nonlinear systems. The function $\phi_n(z)$ can be viewed as the determinant of a tridiagonal matrix, namely:

THEOREM 1. *With the notations as above, we have:*

$$\phi_n(z) = det\left(M_n(z)\right),$$

where

$$\begin{aligned} M_n(z) &= tridiag(z - p_k, p_{k+1} - p_k, p_{k+1} - z)_{0 \leq k < 2n} \\ &= \begin{pmatrix} p_1 - p_0 & p_1 - z & & & \\ z - p_1 & \ddots & \ddots & & \\ & \ddots & \ddots & p_{2n-1} - z & \\ & & z - p_{2n-1} & p_{2n} - p_{2n-1} \end{pmatrix}. \end{aligned}$$

PROOF. Note that $M_n(z)$ has dimensions $2n \times 2n$. We extend the definition of $M_n(z)$ to odd dimensions by posing

$$\hat{M}_m = tridiag(z - p_k, p_{k+1} - p_k, p_{k+1} - z)_{0 \leq k < m},$$

so that $M_n(z) = \hat{M}_{2n}(z)$. Letting $d_m = det(\hat{M}_m(z))$, we get, for $m \geq 4$:

$$\begin{aligned} d_m &= (p_m - p_{m-1})d_{m-1} + (z - p_{m-1})^2 d_{m-2} \\ &= (p_m - p_{m-1})[(p_{m-1} - p_{m-2})d_{m-2} + (z - p_{m-2})^2 d_{m-3}] + (z - p_{m-1})^2 d_{m-2} \\ &= \left[(p_m - p_{m-1})(p_{m-1} - p_{m-2}) + (z - p_{m-1})^2\right] d_{m-2} \\ &\quad + \frac{p_m - p_{m-1}}{p_{m-2} - p_{m-3}} (z - p_{m-2})^2 \left[d_{m-2} - (z - p_{m-3})^2 d_{m-4}\right]. \end{aligned} \tag{6}$$

On the other hand, we have for $n \geq 2$:

$$\varphi_n(z) = \varphi_{n-1}(z) + \frac{1}{z-p_{2n}} - \frac{1}{z-p_{2n-1}} = \varphi_{n-1}(z) + \frac{p_{2n}-p_{2n-1}}{(z-p_{2n})(z-p_{2n-1})}.$$

We obtain, for $z \in \mathbb{C}\backslash\{p_0,\ldots,p_{2n}\}$ ($Q_0(z) = \phi_0(z) = 1$):

$$\begin{aligned} \phi_n(z) &= (z-p_{2n})(z-p_{2n-1})\phi_{n-1}(z) + (p_{2n}-p_{2n-1})P_{n-1}(z)Q_{n-1}(z) \\ &= (p_{2n}-p_{2n-1})\left(P_{n-1}(z)Q_{n-1}(z) - (z-p_{2n-1})\phi_{n-1}(z)\right) \\ &\qquad +(z-p_{2n-1})^2\phi_{n-1}(z) \end{aligned} \tag{7}$$

with

$$\begin{aligned} P_{n-1}(z)Q_{n-1}(z) &= (z-p_{2n-2})(z-p_{2n-3})P_{n-2}(z)Q_{n-2}(z) \\ &= \frac{(z-p_{2n-2})(z-p_{2n-3})}{p_{2n-2}-p_{2n-3}}\left(\phi_{n-1}(z) - (z-p_{2n-2})(z-p_{2n-3})\phi_{n-2}(z)\right) \end{aligned}$$

so that

$$\begin{aligned} &\phi_n(z) - (z-p_{2n-1})^2\phi_{n-1}(z) \\ &= (p_{2n}-p_{2n-1})\left(\left[\frac{(z-p_{2n-2})(z-p_{2n-3})}{p_{2n-2}-p_{2n-3}} - (z-p_{2n-1})\right]\phi_{n-1}(z)\right. \\ &\qquad \left. -\frac{(z-p_{2n-2})^2(z-p_{2n-3})^2}{p_{2n-2}-p_{2n-3}}\phi_{n-2}(z)\right) \\ &= \frac{p_{2n}-p_{2n-1}}{p_{2n-2}-p_{2n-3}}\left(\left[(z-p_{2n-2})^2 + (p_{2n-2}-p_{2n-3})(p_{2n-1}-p_{2n-2})\right]\phi_{n-1}(z)\right. \\ &\qquad \left. -(z-p_{2n-2})^2(z-p_{2n-3})^2\phi_{n-2}(z)\right) \\ &= \frac{p_{2n}-p_{2n-1}}{p_{2n-2}-p_{2n-3}}(z-p_{2n-2})^2\left[\phi_{n-1}(z) - (z-p_{2n-3})^2\phi_{n-2}(z)\right] \\ &\qquad +(p_{2n}-p_{2n-1})(p_{2n-1}-p_{2n-2})\phi_{n-1}(z). \end{aligned} \tag{8}$$

For $m = 2n$, the two recurrence relations (8) satisfied by ϕ_n, ϕ_{n-1} and ϕ_{n-2} and (6) satisfied by d_m, d_{m-1}, and d_{m-2} are thus identical. Since $\phi_0(z) = d_0 = 1$ and $\phi_1(z) = d_2 = (z-p_1)^2 + (p_2-p_1)(p_1-p_0)$, we get $\phi_n(z) = d_{2n} = det\,(M_n(z))$. $\square$

The matrix M_n is typical of finite difference discretizations of certain linear ordinary differential equations. In Section 3, we show that ϕ_n has only complex zeros, and thus, that the matrix $M_n(x)$ is nonsingular for any $x \in \mathbb{R}$.

Note that the induction on d_m can be rewritten as

$$d_m/d_{m-1} = p_m - p_{m-1} + \frac{(z-p_{m-1})^2}{d_m/d_{m-1}},$$

and thus

$$d_m = 0 \quad \text{iff} \quad p_m - p_{m-1} + \cfrac{(z-p_{m-1})^2}{p_{m-1}-p_{m-2} + \cfrac{(z-p_{m-2})^2}{\ldots + \cfrac{(z-p_1)^2}{p_1-p_0}}} = 0.$$

It follows that a root z of $d_{2n} = \phi_n(z) = 0$ is also a solution of the nonlinear system

$$\left\{ \begin{array}{lcl} z_{2n-1}z_{2n-2} & = & (p_{2n} - p_{2n-1})z_{2n-4} + (z - p_{2n-1})^2 \\ \dots & = & \dots \\ z_1 z_0 & = & (p_2 - p_1)z_0 + (z - p_1)^2 \end{array} \right.$$

when setting $z_{2n-1} = 0$ and $z_0 = p_1 - p_0$.

3. Bounding φ_n on the real axis

In this section we derive two lower bounds and an upper bound for the function $|\varphi_n(z)|$ when z is real.

3.1. Two lower bounds. The main result is:

THEOREM 2. *For $n \geq 1$ and $x \in \mathbb{R}\backslash\{p_0, \dots, p_{2n}\}$*

$$|\varphi_n(x)| \geq \psi_n(x) \stackrel{def}{=} \frac{\prod_{k=1}^{n} \left[(x - p_{2k-1})^2 + (p_{2k-1} - p_{2k-2})(p_{2k} - p_{2k-1})\right]}{\prod_{k=0}^{2n} |x - p_k|} \tag{9}$$

PROOF. From Theorem 1, we have, $d_0 = 1 > 0$, $d_1 = p_1 - p_0 > 0$ and

$$d_m = (p_m - p_{m-1})d_{m-1} + (x - p_{m-1})^2 d_{m-2} \qquad \text{for } m \geq 2.$$

Since $p_m - p_{m-1} > 0$, it is clear that all $d_m > 0$. Hence, $d_{m-1} > (p_{m-1} - p_{m-2})d_{m-2}$, and thus

$$d_m > \left[(p_m - p_{m-1})(p_{m-1} - p_{m-2}) + (x - p_{m-1})^2\right] d_{m-2},$$

or

$$\phi_{n+1}(x) \geq \left[(x - p_{2n+1})^2 + (p_{2n+2} - p_{2n+1})(p_{2n+1} - p_{2n})\right] \phi_n(x),$$

i.e.,

$$\phi_n(x) \geq \prod_{k=1}^{n} \left[(x - p_{2k-1})^2 + (p_{2k-1} - p_{2k-2})(p_{2k} - p_{2k-1})\right] > 0 \tag{10}$$

for $x \in \mathbb{R}$. Since

$$\varphi_n(x) = \frac{\phi_n(x)}{P_n(x)Q_n(x)}$$

for $x \in \mathbb{R}\backslash\{p_0, \dots, p_{2n}\}$, the result follows. □

In particular, the function $\varphi_n(z)$ has no real zeros. Comparing φ_n and ψ_n, note that they have identical poles, namely $p_0, \dots, p_{2n}$. From (7), the polynomial function $\phi_n(x)$ has degree $2n$; we conclude that φ_n has n pairs of complex conjugate roots.

Note that the roots of ψ_n actually lie on the circles of center $(p_{2k} + p_{2k-2})/2$ and radius $(p_{2k} - p_{2k-2})/2$, $1 \leq k \leq n$.

An immediate consequence of Theorem 2 is that

COROLLARY 3. *For $n \geq 1$ and $x \in \mathbb{R}\backslash\{p_0, \dots, p_{2n}\}$*

$$|\varphi_n(x)| \geq \frac{\prod_{k=1}^{n} |x - p_{2k-1}|}{\prod_{k=0}^{n} |x - p_{3k}|} = |f_n(x)| \tag{11}$$

This bound on φ_n is not as tight as the previous one, in particular when $x \simeq p_{2j-1}$ for some $1 \le j \le n$. Indeed,

$$\left(\frac{\prod_{k=1}^{n} |x - p_{2k-1}|}{\prod_{k=0}^{n} |x - p_{2k}|} \right)_{x=p_{2j-1}} = 0$$

while $|\varphi_n(p_{2j-1})| = +\infty$ (see Figure 1). The bound (9) is usually much tighter (see Figure 2).

3.2. An upper bound. A simple consequence of the fact that $\phi_n(x) > 0$ for $x \in \mathbb{R}$ is:

THEOREM 4. *For all* $0 \le k < 2n$ *we have:*

$$\forall x \in (p_k, p_{k+1}) \qquad 0 < (-1)^k \varphi_n(x) \le \frac{1}{x - p_k} - \frac{1}{x - p_{k+1}}, \tag{12}$$

PROOF. The left inequality results from (2), (5) and (10).

For i given, the function

$$\varphi_n^*(x) = \sum_{\substack{j=0 \\ j \neq k, k+1}}^{2n} \frac{(-1)^j}{x - p_j}$$

can be written as

$$\varphi_n^*(x) = \frac{\phi_n^*(x)}{P_n^*(x) Q_n^*(x)},$$

where

$$P_n^*(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} (x - p_{2j}), \qquad Q_n^*(z) = \prod_{\substack{j=1 \\ j \neq i+1}}^{n} (z - q_{2j-1}), \qquad \text{where } i = \left\lceil \frac{k}{2} \right\rceil$$

and $\phi_n^*(x) > 0$ for $x \in \mathbb{R}$. Therefore, by (2), $(-1)^k \varphi_n^*(x) < 0$ for $x \in (p_k, p_{k+1})$ (and more generally for $x \in (p_{k-1}, p_{k+2})$), and

$$0 < (-1)^k \varphi_n(x) = (-1)^k \left[\varphi_n^*(x) + \frac{(-1)^k}{x - p_k} + \frac{(-1)^{k+1}}{x - p_{k+1}} \right] < \frac{1}{x - p_k} - \frac{1}{x - p_{k+1}}.$$

□

A numerical comparison between $|\varphi_4|$ from Figures 1 and 2, and the upper bound from Theorem 4

$$\xi(x) = \begin{cases} \dfrac{q - 2.4}{(x - 2.4)(q - x)} & \text{if } x \in (2.4, q) \\ \dfrac{3 - q}{(x - q)(3 - x)} & \text{if } x \in (q, 3) \end{cases}$$

shows the ratio between the two function for the two cases $x \in (2.4, q)$ and $x \in (q, 3)$. The bound of Theorem 4 is usually good when q does not come too close to 2.4 or 3 (i.e., when d is not too small, see Figure 3).

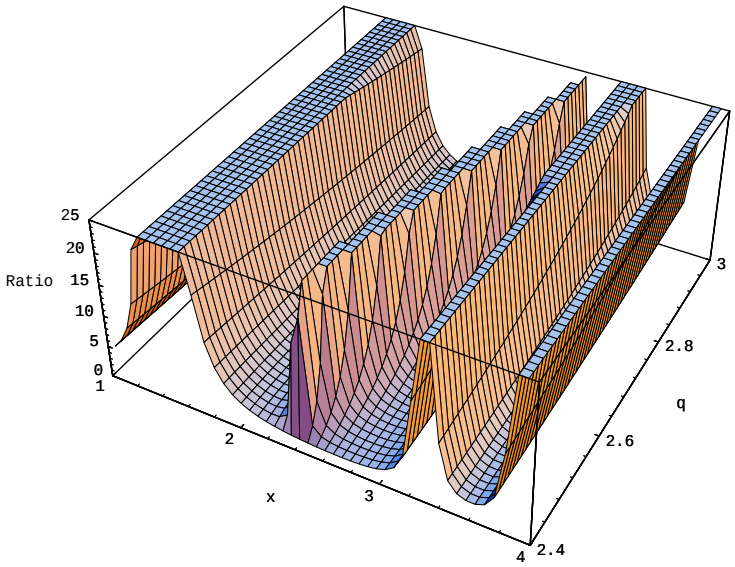

FIGURE 1. Ratio $\dfrac{|\varphi_4(x)|}{|f_4(x)|}$ for $1 \leq x \leq 4$ and $2.4 \leq q \leq 3$, with $(p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (1, 1.5, 2.4, q, 3, 3.3, 3.5, 3.9, 4)$.

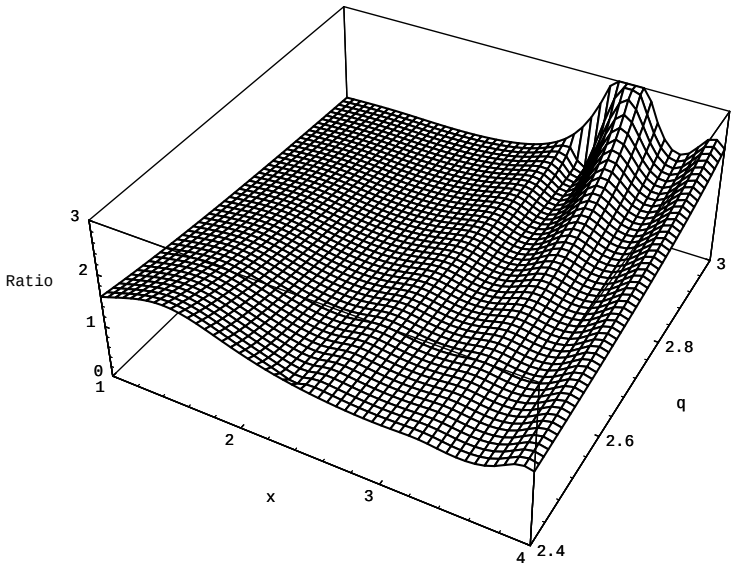

FIGURE 2. Ratio $\dfrac{|\varphi_4(x)|}{|\psi_4(x)|}$ for $1 \leq x \leq 4$ and $2.4 \leq q \leq 3$, with $(p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (1, 1.5, 2.4, q, 3, 3.3, 3.5, 3.9, 4)$. Note that the scale is smaller than for figure 1 above.

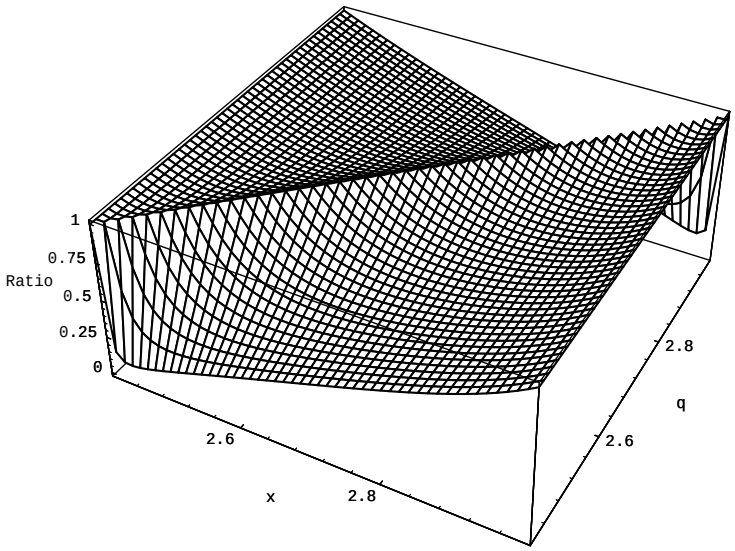

FIGURE 3. Ratio $\dfrac{|\varphi_4(x)|}{|\xi(x)|}$ for $2.4 < x < 3$ and $2.4 < q < 3$, with $(p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (1, 1.5, 2.4, q, 3, 3.3, 3.5, 3.9, 4)$.

4. Zero free regions of φ_n

In the following we will use the notations

$$\mathcal{D}(z_0, r) = \{z \in \mathbb{C}, |z - z_0| < r\}$$

and

$$\mathcal{C}(z_0, r) = \{z \in \mathbb{C}, |z - z_0| = r\}$$

to denote an *open* disk of the complex plane and its boundary, respectively. The closure of a disk $\mathcal{D}(z_0, r)$ will be denoted by $\overline{\mathcal{D}}(z_0, r)$.

Because of the equivalences

$$\begin{aligned}\varphi_n(z_0) = 0 &\iff (\Re e\,(\varphi_n(z_0)) = 0 \text{ and } \Im m\,(\varphi_n(z_0)) = 0)\\ &\iff \left(\Re e\,(\varphi_n(z_0)) = 0 \text{ and } \left|\sum_{k=0}^{n} \frac{1}{z - p_{2k}}\right| = \left|\sum_{k=1}^{n} \frac{1}{z - p_{2k-1}}\right|\right),\end{aligned}$$

it is easy to verify that a root z_0 of φ_n satisfies

$$\sum_{k=1}^{n} \frac{1}{a_{2k}^2} = \sum_{k=1}^{n} \frac{1}{a_{2k-1}^2}, \tag{13}$$

$$\sum_{k=1}^{n} \frac{p_{2k}}{a_{2k}^2} = \sum_{k=1}^{n} \frac{p_{2k-1}}{a_{2k-1}^2}, \tag{14}$$

$$\sum_{1 \le k < l \le n} \frac{\cos\theta_{2k,2l}}{a_{2k}a_{2l}} = \sum_{1 \le k < l \le n} \frac{\cos\theta_{2k-1,2l-1}}{a_{2k-1}a_{2l-1}}, \tag{15}$$

where $a_i = |z_0 - p_i|, \theta_{i,j}$ is the angle between $\overrightarrow{z_0 p_i}$ and $\overrightarrow{z_0 p_j}$, for $0 \leq i, j \leq 2n$ (see Figure 4). In the case $n = 1$ the relation (15) implies $\cos\theta_{0,2} = 0$, i.e., z_0 is on the circle $\mathcal{C}((p_0 + p_2)/2, (p_2 - p_0)/2)$.

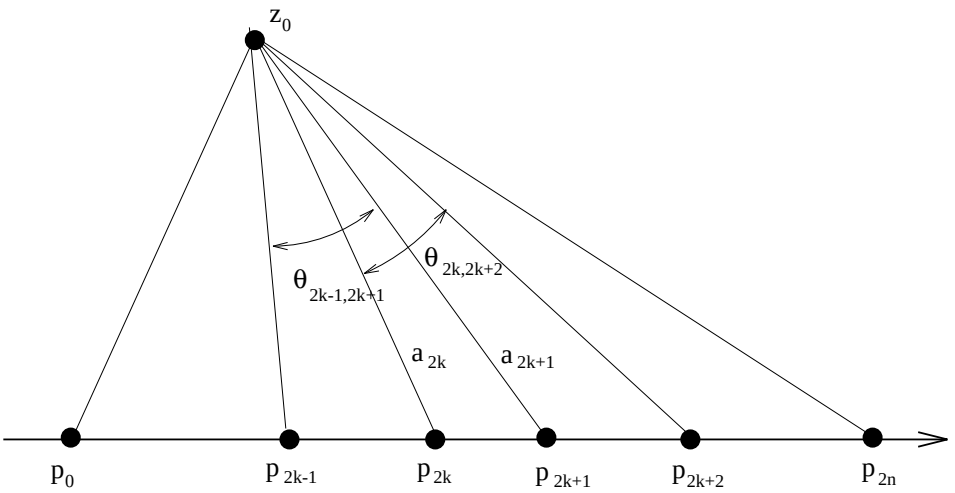

FIGURE 4. Geometric interpretation

We show that the zeros of φ_n are located in the disk of center $\dfrac{p_0 + p_{2n}}{2}$ and radius $\dfrac{p_{2n} - p_0}{2}$, but that none of these roots lie in the open disks of center $\dfrac{p_{k+2} + p_k}{2}$ and radius $\dfrac{p_{k+2} - p_k}{2}$ for any $1 \leq k \leq 2n - 2$. In terms of (15), this is equivalent to

$$\cos\theta_{k,k+2} \quad \geq 0 \qquad \text{for } 1 \leq k \leq 2n - 2$$

and

$$\cos(\theta_{0,2n}) < 0.$$

Unfortunately, although the proof is straightforward in the case $n = 2$, we could not show the result directly, based on (13), (14) and (15). Instead, the proofs presented here were found by studying the problem geometrically, and the reader is encouraged to think in those terms.

We first formally state the result:

THEOREM 5. *If* $\varphi_n(z_0) = 0$, *then*

$$z_0 \notin \mathcal{D}\left(\frac{p_{k+2} + p_k}{2}, \frac{p_{k+2} - p_k}{2}\right) \qquad \textit{for } 0 \leq k \leq 2n - 2, \tag{16}$$

$$z_0 \in \overline{\mathcal{D}}\left(\frac{p_{2n} + p_0}{2}, \frac{p_{2n} - p_0}{2}\right) \tag{17}$$

and

$$p_1 \leq \Re e(z_0) \leq p_{2n-1}. \tag{18}$$

Figure 5 illustrates the case $n = 2$, for specific values of p_0, p_2, p_3 and p_4. Here p_1 varies from p_0 to p_2. In the limiting cases of $p_1 = p_0$ or p_2 the problem reduces to the case $n = 1$, for which the zeros are

$$z = p_3 \pm \imath\sqrt{(p_3 - p)(p_4 - p_3)},$$

with $p = p_0$ if $p_1 = p_2$, and $p = p_2$ if $p_1 = p_0$. Note that the shaded area corresponds to zero-free regions depending on p_0, p_2, p_4 only, since p_1 is varying here. One can easily verify that $a \leq \Re e(z_1) \leq p_3$ (the second inequality being clear from Figure 5).

As p_1 increases, the first pair of complex roots remains outside the disk $\overline{\mathcal{D}}(2, 1)$ (but quite close), while the second pair gets farther from the real axis. Both pairs remain inside the disk $\mathcal{D}(2.75, 1.75)$. Although limited to 5 points, this example is typical of the behavior of the roots of φ_n.

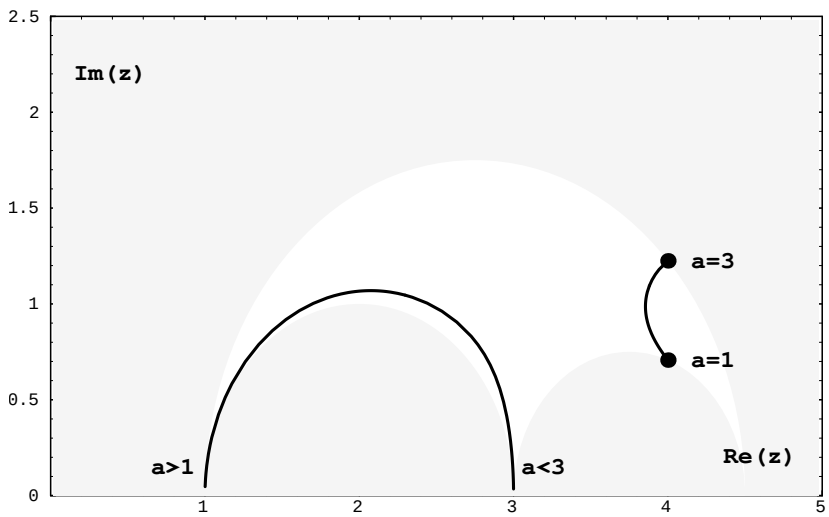

FIGURE 5. The two lines indicate the location of the two zeros of $\varphi_2(z)$ in the upper complex plane, for $(p_0, p_1, p_2, p_3, p_4) = (1, a, 3, 4, 4.5)$ with $1 < a < 3$. When $a = 1$ or $a = 3$, $\varphi_2(z)$ simplifies and the black dots represent the corresponding roots z_0.

The proof of (16) and (17) is based on transforming the algebraic problem into a geometrical one via a standard conformal mapping. We start with a few definitions and lemmas which will help us prove Theorem 5.

LEMMA 6. *For all* $a, b, c \in \mathbb{R}$,

$$|e^{\imath a} - e^{\imath b} + e^{\imath c}|^2 - 1 = 8 \sin\left(\frac{a-b}{2}\right) \sin\left(\frac{c-b}{2}\right) \cos\left(\frac{a-c}{2}\right).$$

PROOF. Once the left hand side is expanded, this is a straightforward exercise with trigonometric identities. □

DEFINITION 7. *Let $R_{a,b}(z_0)$ denote the reflection of a point z_0 over the line containing the two points a and b. Explicitly,*

$$R_{a,b}(z_0) = \frac{\bar{a}b - a\bar{b} + \bar{z_0}(a-b)}{\bar{a} - \bar{b}}.$$

DEFINITION 8. *A closed arc of a circle $\overset{\frown}{ad}$ is called a* minor arc *if it comprises no more than half the circle.*

Note that for a given arc $\overset{\frown}{ad}$, there is a natural ordering on the arc for which $a < d$.

Next we consider all vectors with both head and tail on an arc $\overset{\frown}{ad}$ (with head near d and tail near a), and translate them so that their tail are at a. The set described by the new position of their head is identified geometrically in the following Lemma:

LEMMA 9. *For $z_0 \in \mathbb{C}, r > 0$ and any minor arc $\overset{\frown}{ad}$ on the circle $\mathcal{C}(z_0, r)$, let*

$$S_{a,d} = \{z \in \mathbb{C} | \ \text{there exist } b, c \in \overset{\frown}{ad} \ \text{ s.t. } a \le b \le c \le d \text{ and } z - a = c - b\}.$$

Then,

$$S_{a,d} = \overline{\mathcal{D}}(z_0, r) \bigcap \overline{\mathcal{D}}(R_{a,d}(z_0), r).$$

PROOF. It is easy to check that the statement of the lemma is homogeneous under translation and scalar multiple, and is invariant under rotation of center z_0. So, we may assume, without loss of generality, that $\mathcal{C}(z_0, r) = \mathcal{C}(0, 1)$,

$$d = e^{\imath\theta} \text{ and } a = e^{-\imath\theta} \text{ for some } 0 < \theta \le \pi/2$$

(see Figure 6).

Let

$$b = e^{\imath\psi_1} \text{ and } c = e^{\imath\psi_2} \text{ with } -\theta \le \psi_1 \le \psi_2 \le \theta.$$

We first show that $S_{a,d} \subset \overline{\mathcal{D}}(0, 1)$. If $z \in S_{a,d}$, then

$$z = c - b + a = e^{\imath\psi_2} - e^{\imath\psi_1} + e^{-\imath\theta}.$$

To check that $z \in \overline{\mathcal{D}}(0, 1)$, note that by Lemma 6

$$|e^{\imath\psi_2} - e^{\imath\psi_1} + e^{-\imath\theta}|^2 - 1 = 8 \sin\left(\frac{\psi_2 - \psi_1}{2}\right) \sin\left(\frac{-\theta - \psi_1}{2}\right) \cos\left(\frac{\psi_2 + \theta}{2}\right).$$

From the bounds on ψ_1, ψ_2, and θ we see that

$$\begin{aligned} 0 &\le \frac{\psi_2 - \psi_1}{2} \le \pi/2, \\ -\pi/2 &\le \frac{-\theta - \psi_1}{2} \le 0 \end{aligned}$$

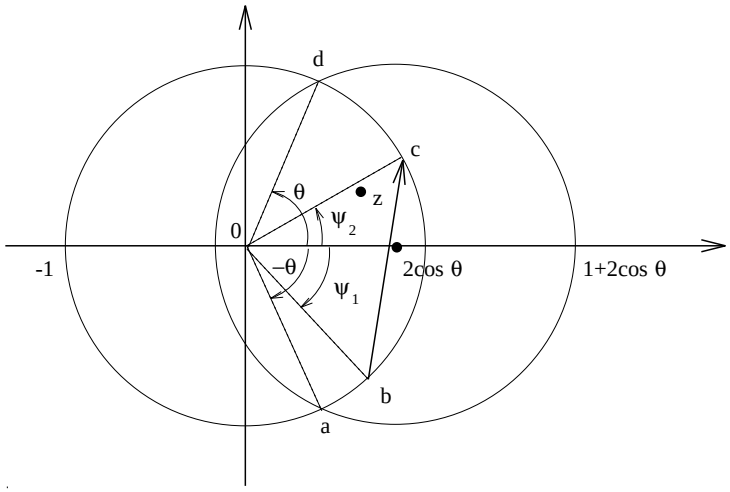

FIGURE 6. $S_{a,d}$.

and

$$0 \quad \leq \quad \frac{\psi_2 + \theta}{2} \leq \pi/2.$$

So, $|z|^2 - 1 \leq 0$, which implies that $z \in \overline{\mathcal{D}}(0,1)$.

Now, $R_{a,d}(0) = 2\cos\theta$. So, to see that $z \in \overline{\mathcal{D}}(2\cos\theta, 1)$, we must show that

$$|e^{\imath\psi_2} - e^{\imath\psi_1} + e^{-\imath\theta} - 2\cos\theta| \leq 1.$$

But,

$$\begin{aligned} |z - 2\cos\theta| &= |e^{\imath\psi_2} - e^{\imath\psi_1} + e^{-\imath\theta} - 2\cos\theta| \\ &= |-e^{-\imath\psi_2} + e^{-\imath\psi_1} - e^{\imath\theta} + 2\cos\theta| \\ &= |-e^{-\imath\psi_2} + e^{-\imath\psi_1} + e^{-\imath\theta}| \end{aligned}$$

Now, we have the inequalities

$$-\pi/2 \leq -\theta \leq -\psi_2 \leq -\psi_1 \leq \theta \leq \pi/2,$$

so that $|z - 2\cos\theta|^2 - 1 < 0$ by applying Lemma 6. Therefore,

$$S_{a,d} \subset \overline{\mathcal{D}}(z_0, r) \bigcap \overline{\mathcal{D}}(R_{a,d}(z_0), r).$$

The reverse inclusion follows from topological considerations. Consider the continuous function $G\colon A \to \mathbb{C}$ defined by

$$G(\psi_1, \psi_2) = e^{\imath\psi_2} - e^{\imath\psi_1} + e^{-\imath\theta},$$

where A is the triangle in $\mathbb{R}^2$ described by

$$-\theta \leq \psi_1 \leq \psi_2 \leq \theta.$$

Note that $S_{a,d}$ is the image of G. From the first part of the proof of this lemma, we know that the image of G is contained in $\boldsymbol{D} = \overline{\mathcal{D}}(z_0, r) \bigcap \overline{\mathcal{D}}(R_{a,d}(z_0), r)$. To see

that G is surjective, we need only check that the boundary of A is mapped to the boundary of $\boldsymbol{D}$, and that the winding number of the boundary (viewed as a loop about any point in the interior) is preserved up to sign. This is straightforward. □

Remark: Note that $S_{a,d}$ is invariant under the transformation $z \mapsto a + d - z$.

LEMMA 10 (Reduction Step). *Given a minor circular arc $\overset{\frown}{ad}$ containing points b and c so that $a < b < c < d$, there exists a pair of points e and f on the arc such that $a < e < f < d$ and $(b-a)+(d-c) = f - e$. Moreover, $e < c$ and $f > b$.*

PROOF. The existence of e and f is equivalent to showing that $b + d - c \in S_{a,d}$. First we observe that $b - a + d - c = d - a - (c - b)$. By definition, $c - b + a \in S_{a,d}$. From the remark, we then have that $b + d - c = a + d - (c - b + a) \in S_{a,b}$ as desired.

The inequalities $e < c$ and $f > b$ follow because $\overset{\frown}{ad}$ is a minor arc. In particular, the ordering on $\overset{\frown}{ad}$ can be induced by projecting onto the vector $\vec{ad}$. If we write $comp_{\vec{ad}}\vec{x}$ for the $\vec{ad}$ component of a vector $\vec{x}$, we have

$$comp_{\vec{ad}}\vec{ab} + comp_{\vec{ad}}\vec{cd} = comp_{\vec{ad}}\vec{ef}.$$

As each of these terms is positive, we obtain

$$comp_{\vec{ad}}\vec{cd} \leq comp_{\vec{ad}}\vec{ef} \leq comp_{\vec{ad}}\vec{ed}$$

and

$$comp_{\vec{ad}}\vec{ab} \leq comp_{\vec{ad}}\vec{ef} \leq comp_{\vec{ad}}\vec{af}.$$

In particular, $e < c$ and $f > b$. □

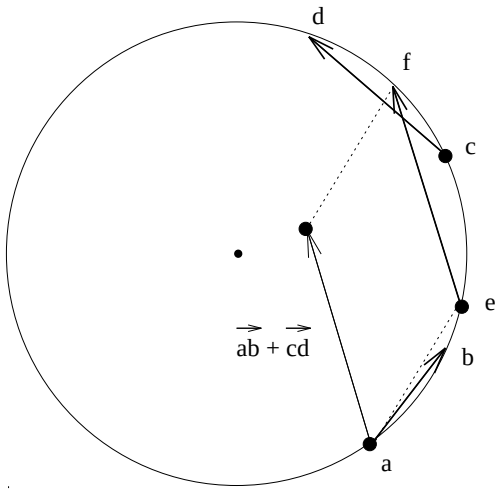

FIGURE 7. $\overrightarrow{ab} + \overrightarrow{cd} = \overrightarrow{ef}$.

Lemma 10 can be interpreted as follows: given two disjoint minor arcs $\overset{\frown}{ab}$ and $\overset{\frown}{cd}$ included in a minor arc $\overset{\frown}{ad}$, there exist a minor arc $\overset{\frown}{ef}$ included in $\overset{\frown}{ad}$ such that

$$\overrightarrow{ab} + \overrightarrow{cd} = \overrightarrow{ef}$$

(see Figure 7).

We can now prove Theorem 5.

PROOF OF THEOREM 5. We first suppose that there exists $n \geq 1$, a set of $\{p_0, \ldots, p_{2n}\}$, and a root $z_0 \in \mathbb{C}$ which provide a counterexample for either (16) with k even, or (17), i.e., we assume that either

Case I: $\exists z_0 \in \mathbb{C}$ such that $\varphi_n(z_0) = 0$ and $z_0 \in \mathcal{D}\left(\frac{p_{k+2}+p_k}{2}, \frac{p_{k+2}-p_k}{2}\right)$ for some $0 \leq k \leq 2n-2$, k even;

or

Case II: $\exists z_0 \in \mathbb{C}$ such that $\varphi_n(z_0) = 0$ and $z_0 \notin \overline{\mathcal{D}}\left(\frac{p_{2n}+p_0}{2}, \frac{p_{2n}-p_0}{2}\right)$.

Note that $z_0 \notin \mathbb{R}$ by Theorem 2. The original problem is homogeneous under addition of a real scalar and multiplication by a non-zero real number. So, we may assume $z_0 = \imath/2$.

Consider the functional mapping

$$g(z) = \frac{2}{\imath - 2z}.$$

Some of the properties of g can be easily verified:

- g is a fractional linear transformation sending complex circles (i.e., lines or circles) to complex circles in $\mathbb{C}$ (as a map from $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to $\hat{\mathbb{C}}$);
- g preserves angles (there are no ramification points);
- the image under g of the extended real axis is a circle $\mathcal{C}$ of radius 1, tangent to the real axis at the origin, located in the lower half of the complex plane, i.e.,
$$\mathcal{C} = g(\mathbb{R} \cup \infty) = \mathcal{C}(-\imath, 1);$$

For $0 \leq j \leq 2n$ we define $p_j^* = g(p_j)$. The equality $\varphi_n(z_0) = 0$ thus yields

$$0 = \sum_{j=0}^{n} p_{2j}^* - \sum_{j=1}^{n} p_{2j-1}^*. \tag{19}$$

Note that the points p_j^* lie on $\mathcal{C} \setminus \{0\}$, and that their ordering on $\mathcal{C}$ is the same as for the p_j in $\mathbb{R}$ (where left to right on the real axis corresponds to clockwise on the circle).

We describe the positions of the p_j^*'s on the circle in each of the two cases (16) and (17) (referred to as Case I and Case II, respectively):

Case I: The circle C' passing through $\imath/2$ and centered at $(p_{k+2}+p_k)/2$ is in the interior of $\mathcal{D}\left((p_{k+2}+p_k)/2,(p_{k+2}-p_k)/2\right)$. Let $\mathcal{L}=g(C')$. Since C' contains $\imath/2$, $\mathcal{L}$ is a straight line. Since C' is centered on the real axis, it intersects the real axis at right angles. Hence $\mathcal{L}$ intersects the circle C at right angles as well, that is, it passes through the center of C.
Now, 0 lies inside the circle C' and all the p_j with $j\neq k+1$ lie outside the circle (along with ∞). So, in the image plane, $-2\imath=g(0)$ lies on one side of the line $\mathcal{L}$, and the p_j^*, (for $j\neq k+1$) and $0=g(\infty)$ lie on the other (see Figure 8).

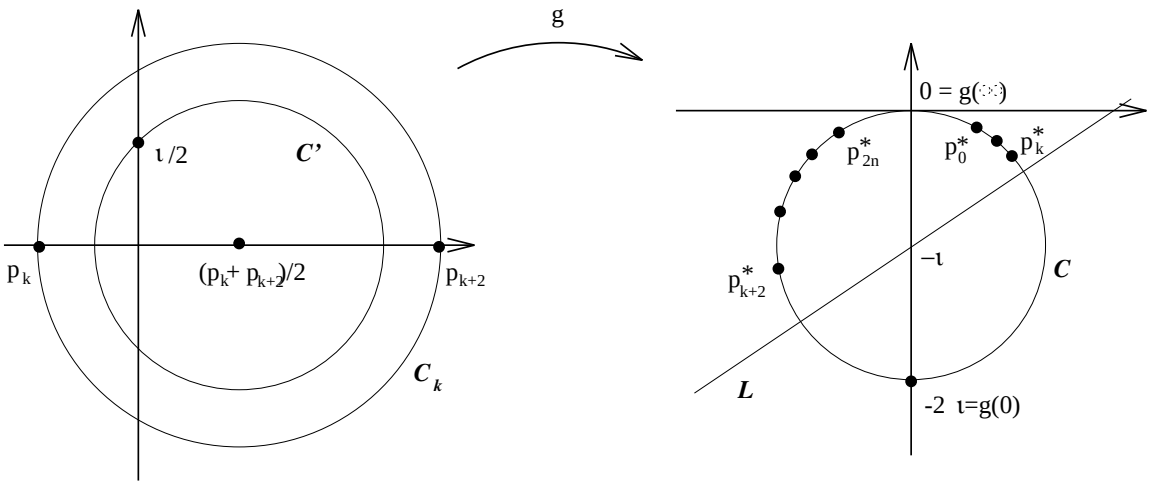

FIGURE 8. Case I.

Case II: Consider the circle C' through $\imath/2$ centered at $(p_0+p_{2n})/2$. Again, we let $\mathcal{L}=g(C')$ which must be a line through the center of C. In this case, the fact that

$$\frac{\imath}{2}\notin\overline{\mathcal{D}}\left(\frac{p_{2n}+p_0}{2},\frac{p_{2n}-p_0}{2}\right)$$

implies that 0 and all of the p_j lie inside of C'. Hence, $g(0)$ and all of the p_j^* lie on one side of $\mathcal{L}$, and $0=g(\infty)$ lies on the other side (see Figure 9).

Note that in both cases, all p_j^*'s (with the possible exception of p_{k+1}^* in Case I) are on a minor arc of $\mathcal{C}$.

Since sums of complex numbers corresponds to vector addition, the problem is now reduced to a question of geometry. For example, the differences $p_{j+1}^*-p_j^*$ correspond to vectors with head and tail on the circle $\mathcal{C}$. We introduce $p_{2n+1}^*=0$.

Case I: We have,

$$p_1^*-p_0^*+\ldots+p_{k-1}^*-p_{k-2}^*+p_{k+3}^*-p_{k+2}^*+\ldots-p_{2n}^*+p_{2n+1}^*=-(p_{k+1}^*-p_k^*),$$

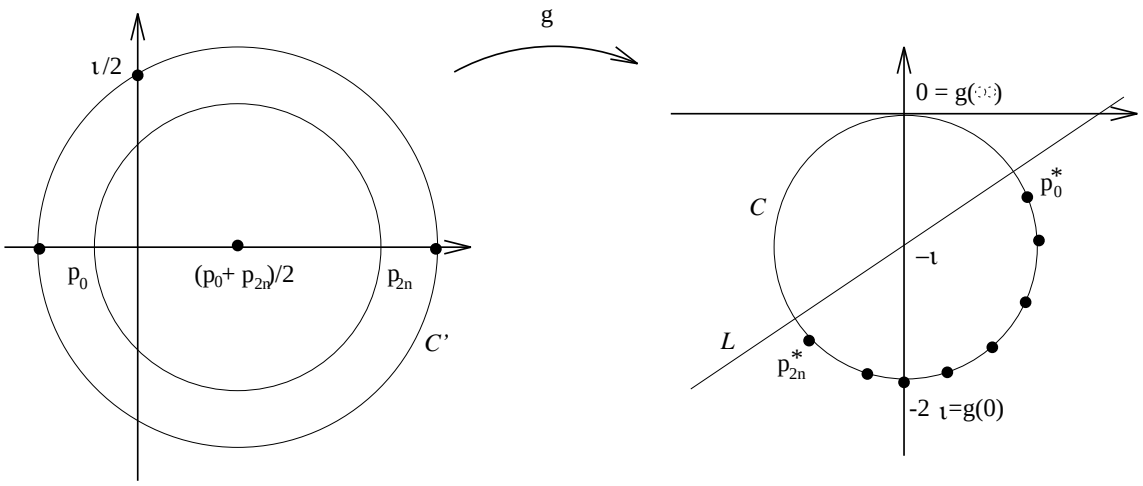

FIGURE 9. Case II.

i.e.,

$$\overrightarrow{p^*_{k+3}p^*_{k+2}} + \ldots + \overrightarrow{p^*_{2n}p^*_{2n+1}} + \overrightarrow{p^*_0p^*_1} + \ldots + \overrightarrow{p^*_{k-2}p^*_{k-1}} = -\overrightarrow{p^*_kp^*_{k+1}} .$$

By Lemma 10, since $p^*_{k+3}, \ldots, p^*_{2n}, 0, p^*_0, \ldots, p^*_{k-1}$ lie on a minor arc of $\mathcal{C}$ (namely on one side of $\mathcal{L}$), there exist p^* and q^* on $\mathcal{C}$ such that

$$\overrightarrow{p^*q^*} + \overrightarrow{p^*_kp^*_{k+1}} = 0$$

and $p^* < q^* < p^*_k$ lie on a minor arc.

Case II: Similarly,

$$\overrightarrow{p^*_0p^*_1} + \ldots + \overrightarrow{p^*_{2n-2}p^*_{2n-1}} = -\overrightarrow{p^*_{2n}p^*_{2n+1}} .$$

By Lemma 10 again, since $p^*_0, \ldots, p^*_{2n-1}$ lie on a minor arc of $\mathcal{C}$ (namely on one side of $\mathcal{L}$), there exist p^* and q^* on $\mathcal{C}$ such that

$$\overrightarrow{p^*q^*} + \overrightarrow{p^*_{2n}p^*_{2n+1}} = 0$$

and $p^* < q^* < p^*_{2n}$ lie on a minor arc.

The fact that the two cases lead to a similar conclusion is not surprising, since the exterior of the disk $\overline{\mathcal{D}}((p_{2n}+p_0)/2, (p_{2n}-p_0)/2)$ can be identified with the interior of a disk on the Riemann sphere containing ∞, and passing through p_0 and p_{2n}.

In order to parameterize the circle $\mathcal{C}$, it is convenient to first normalize it by composing with the isometry $h(z) = \dfrac{1-\imath z}{1-\imath p^*_j}$ ($j = k$ in Case I, $j = 2n$ in Case II), mapping $\mathcal{C}$ to the unit circle $\mathcal{C}(0,1)$.

We have $h(p^*_j) = 1, h(-\imath) = 0$, and we define $p^{**} = h(p^*) = e^{\imath\theta_2}, q^{**} = h(q^*) = e^{\imath\theta_1}$, and finally $p^{**}_{j+1} = h(p^*_{j+1}) = e^{\imath\theta_3}$, for some $\theta_i, i = 1, 2, 3$ (see Figure 10). The ordering of $p^{**}, q^{**}, 1$ and p^{**}_{j+1} on $\mathcal{C}(0,1)$ is the same as the ordering of p^*, q^*, p^*_j and p^*_{j+1} on $\mathcal{C}$. (The map h corresponds to a translation of vector $\imath$ composed with

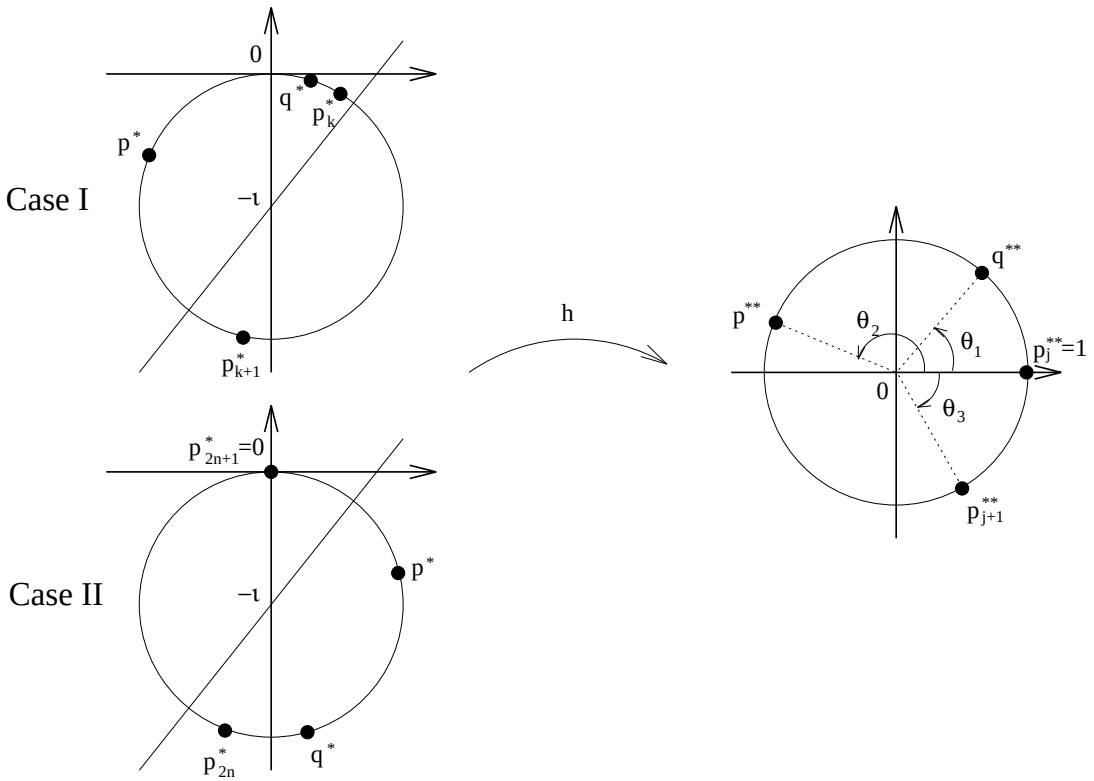

FIGURE 10. The image of $\mathcal{C}$ under the isometry h defined by $h(z) = (1 - \imath z)/(1 - \imath p_j^*)$, where $j = k$ in Case I and $j = 2n$ in Case II.

a rotation of angle $-arg(p_j^* + \imath)$). The condition that $1, q^{**}$ and p^{**} lie within a semi-circle implies that

$$0 < \theta_1 < \theta_2 < \pi.$$

Moreover,

$$\theta_2 - 2\pi < \theta_3 < 0 \quad (-\pi < \theta_3 < 0 \text{ in case II}).$$

The equation $\overrightarrow{p^*q^*} + p_j^*\overrightarrow{p_{j+1}^*} = 0$ is then equivalent to

$$e^{\imath\theta_1} - e^{\imath\theta_2} + e^{\imath\theta_3} = 1.$$

By Lemma 6 we have

$$|e^{\imath\theta_1} - e^{\imath\theta_2} + e^{\imath\theta_3}|^2 - 1 = 8 \sin\left(\frac{\theta_1 - \theta_2}{2}\right) \sin\left(\frac{\theta_3 - \theta_2}{2}\right) \cos\left(\frac{\theta_1 - \theta_3}{2}\right) = 0;$$

the bounds on the angles imply that $\theta_1 = \theta_3 + \pi$, and thus $e^{\imath\theta_1} + e^{\imath\theta_3} = 0$ and $e^{\imath\theta_2} = -1$, contradicting the bound on θ_2. This proves (16) and (17).

The proof of (16) with k odd, and of (18) is an application of (16) with k even

and of (17). Indeed, introduce $p_{-1} = -\infty$ and $p_{2n+1} = +\infty$. Then,

$$\varphi_n(z_0) = 0 \Longleftrightarrow \sum_{j=-1}^{2n+1} \frac{1}{z_0 - p_j} = 0, \tag{20}$$

with

$$p_{-1} < p_1 < p_2 < \ldots < p_{2n-1} < p_{2n} < p_{2n+1},$$

and (16) for k odd results immediately from (16) for k even. Moreover, the disks $\mathcal{D}((p_1 + p_{-1})/2, (p_1 - p_{-1})/2)$ and $\mathcal{D}((p_{2n+1} + p_{2n-1})/2, (p_{2n+1} - p_{2n-1})/2)$ reduce to half-planes, $\{z, \Re e(z) < p_1\}$ and $\{z, \Re e(z) > p_{2n-1}\}$, respectively. Thus, (17) applied to (20) yields (18) when combined. $\square$

Recall that $d = \min_{0 \le i \le 2n-1} (p_{i+1} - p_i)$ is the minimal interval length of the partition of the interval $[p_0, p_{2n}]$ given by the p_j's and q_m's. Then, as a consequence of Theorem 5, we have:

COROLLARY 11. *If* $\varphi(z_0) = 0$, *then* $|\Im m(z_0)| \ge \frac{\sqrt{3}}{2} d$.

PROOF. We can bound $|\Im m(z_0)|$ using the zero-free regions from Theorem 5. A lower bound is given by the intersection points of the circles

$$\mathcal{C}\left(\frac{p_{k+2} + p_k}{2}, \frac{p_{k+2} - p_k}{2}\right) \qquad \text{and} \qquad \mathcal{C}\left(\frac{p_{k+1} + p_{k-1}}{2}, \frac{p_{k+1} - p_{k-1}}{2}\right) \tag{21}$$

The minimum could conceivably occur at the intersection of a circle and the vertical line $\Re e(z) = p_1$ or $\Re e(z) = p_{2n-1}$, but this situation can be handled in a similar but more simple manner (or as a limiting case of the situation treated). We can treat both cases of the intersection of two circles simultaneously as follows.

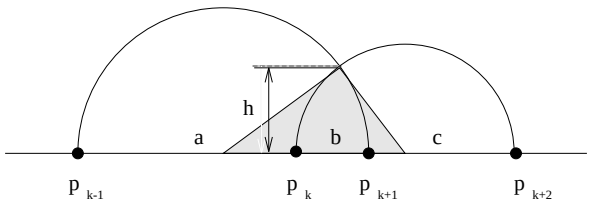

FIGURE 11. h(a,b,c)

We start with two intersecting circles with a, b, c and h as shown in Figure 11, (i.e., $a = p_k - p_{k-1}$, $b = p_{k+1} - p_k$, and $c = p_{k+2} - p_{k+1}$). We need to show that if

$$a, b, c \ge d, \tag{22}$$

then $h \ge d\frac{\sqrt{3}}{2}$.

The shaded triangle of Figure 11 has sides of length $\frac{a+b}{2}$, $\frac{b+c}{2}$, and $\frac{a+b}{2} + \frac{b+c}{2} - b = \frac{a+c}{2}$ (the base). Its area is on one hand given by

$$A = \frac{1}{2} h(a, b, c) \frac{a + c}{2}. \tag{23}$$

On the other hand, it is given by Heron's formula

$$A = \sqrt{s(s - \frac{a+b}{2})(s - \frac{b+c}{2})(s - \frac{a+c}{2})} \tag{24}$$

with $s = \frac{1}{2}\left(\frac{a+b}{2} + \frac{b+c}{2} + \frac{a+c}{2}\right) = \frac{a+b+c}{2}$. Combined with equation 23, we find that

$$h(a,b,c) = \frac{\sqrt{abc}\sqrt{a+b+c}}{a+c} \tag{25}$$

So, we treat this as a minimization problem with the inequality constraints (22).

It is easy to see that for given a and c, h is an increasing function of b. So, we take $b = d$. Then,

$$\frac{\partial h}{\partial a}(a,d,c) = \frac{cd\left(cd + c^2 + a(c-d)\right)}{2\sqrt{acd}\sqrt{a+c+d}(a+c)^2}$$

So, h is an increasing function of a, and we take $a = d$. By the symmetry between a and c, we also get $c = d$. So,

$$h(a,b,c) \geq h(d,d,d) = d\frac{\sqrt{3}}{2}.$$

□

The bound is optimal in some cases. The simplest situation is the case $n = 2$, and $p_1 - p_0 = p_2 - p_1 = p_3 - p_2 = d$, and $p_4 >> p_3$. Then φ_3 has two complex conjugate roots with imaginary parts which approach the bound as p_4 goes to infinity. See also Section 5.

5. Variation of the poles

In this section we consider a variation of the p_k in the definition of $\varphi_n(z)$. We show that as a p_k approaches p_{k+1}, a root of $\varphi_n(z)$ approaches the real axis. In particular, the function $\varphi_n(z)$ can have roots arbitrarily close to the real axis.

To study the variation, we consider a homotopy of the points p_k. So, we fix n and let $p_k(t)$ be continuous functions $[0,1] \to \mathbb{R}$ for $0 \leq k \leq 2n$. We further require that

$$p_k(t) < p_{k+1}(t) \quad \text{for } 0 \leq k < 2n \text{ and for } t \in (0,1]. \tag{26}$$

We let

$$\varphi_{n,t}(z) = \sum_{k=0}^{2n} \frac{(-1)^k}{z - p_k(t)}.$$

Due to the continuity of the $p_k(t)$, we still have the inequalities

$$p_k(0) \leq p_{k+1}(0) \quad \text{for } 0 \leq k < 2n.$$

THEOREM 12. *With the notation as above, if there exists an integer* $0 \leq k^* < 2n$ *such that*

$$p_{k^*}(0) = p_{k^*+1}(0),$$

then at least two roots of $\varphi_{n,t}(z)$ *have limit* $p_{k^*}(0)$ *as* $t \to 0$.

PROOF. We write $\varphi_{n,t}(z)$ as a rational function of z whose coefficients are continuous functions of t by combining the partial fractions to get

$$\varphi_{n,t}(z) = \frac{\phi_{n,t}(z)}{\psi_{n,t}(z)},$$

where

$$\psi_{n,t}(z) = \prod_{k=0}^{2n}(z - p_k(t))).$$

Moreover, $\phi_{n,t}(z)$ is a monic polynomial of degree $2n$ for each t. Note that the fraction $\phi_{n,t}(z)/\psi_{n,t}(z)$ may not be in lowest terms for every $t \in [0,1]$. However, by counting the poles of $\varphi_{n,t}(z)$ for a given value of t, we see that $\phi_{n,t}(z)$ and $\psi_{n,t}(z)$ are relatively prime for all $t \in (0,1]$.

We may consider the roots of $\varphi_{n,t}(z)$ as continuous functions of t. For $t \in (0,1]$, these are exactly the same as the roots of $\phi_{n,t}(z)$. We are interested in the limit of these roots as $t \to 0$. But, since $\phi_{n,t}(z)$ is a monic polynomial of fixed degree for all $t \in [0,1]$, the limits of the roots are simply the roots of $\phi_{n,0}(z)$.

Now, since $p_{k^*}(0) = p_{k^*+1}(0)$ their corresponding terms in the definition of $\varphi_{n,0}(z)$ can be canceled before combining the partial fractions to form

$$\varphi_{n,0}(z) = \frac{\phi_n^*(z)}{\psi_n^*(z)}$$

with $\phi_n^*(z)$ and $\psi_n^*(z)$ relatively prime. But now,

$$\frac{\phi_{n,0}(z)}{\psi_{n,0}(z)} = \frac{\phi_n^*(z)}{\psi_n^*(z)} \qquad \text{for all } z \in \mathbb{C} \setminus \{p_1(0), \ldots, p_{2n}(0)\}.$$

By comparing the denominators of both sides, we see that

$$\frac{\phi_{n,0}(z)}{\phi_n^*(z)} \qquad \text{is divisible by} \qquad (z - p_{j^*}(0))^2.$$

So $\phi_{n,0}(z)$ has at least a double root at $p_{k^*}(0)$. Thus, there are at least two roots of $\varphi_{n,t}(z)$ with limit $p_{k^*}(0)$ as $t \to 0$. $\square$

Theorem 12 shows that when $p_{k+1} \to p_k$ for some $0 \le k \le 2n$, at least two (conjugate) roots tend to p_k as well. The next result provides an estimate on how these roots tend to their limit. We choose $p_k(t) = p_k$ for $0 \le k \ne k^* + 1 \le 2n$, and $p_{k^*+1}(t) = p_{k^*} + t$.

THEOREM 13. *If there exists an integer* $0 \le k^* < 2n$ *such that*

$$p_{k^*}(0) = p_{k^*+1}(0) = p_{k^*},$$

then there exists $z_0 \in \mathbb{C} \setminus \mathbb{R}$ *such that* $\varphi_{n,t}(z_0) = 0$ *and*

$$z_0 = p_{k^*} \pm \imath\, C_1\sqrt{t} + C_2 t + o(t), \tag{27}$$

where

$$C_1 = \frac{1}{\sqrt{|\varphi_n^*(p_{k^*})|}}, \quad C_2 = \frac{1}{2}\left(1 - \frac{\varphi_n^{*\prime}(p_{k^*})}{\varphi_n^*(p_{k^*})|\varphi_n^*(p_{k^*})|}\right),$$

and

$$\varphi_n^*(z) = \sum_{\substack{j=0 \\ j\neq k^*,k^*+1}}^{2n} \frac{(-1)^j}{z-p_j}.$$

PROOF. By Theorem 12, we know $z_0 \to p_{k^*}$, and let $z_0 = p_{k^*}+\xi$. Since $p_{k^*+1}(t) = p_{k^*} + t$ $(t>0)$ we get

$$\begin{aligned} 0 &= \varphi_{n,t}(z_0) \\ &= \frac{(-1)^{k^*}}{\xi} + \frac{(-1)^{k^*+1}}{\xi - t} + \varphi_n^*(z_0) \\ &= (-1)^{k^*+1}\frac{t}{\xi(\xi-t)} + \varphi_n^*(p_{k^*}) + \xi\varphi_n^{*'}(p_{k^*}) + \frac{\xi^2}{2}\varphi_n^{*''}(p_{k^*}) + o(\xi^2). \end{aligned}$$

We then verify that $\xi = \pm\imath\, C_1\sqrt{t} + C_2 t + o(t)$ is a root of the previous equation: the condition $\varphi_{n,t}(z_0) = 0$ leads to

$$\begin{aligned} (-1)^{k^*}t &= \left(\xi^2 - t\xi\right)\left(\varphi_n^*(p_{k^*}) + \xi\varphi_n^{*'}(p_{k^*}) + \frac{\xi^2}{2}\varphi_n^{*''}(p_{k^*}) + o(\xi^2)\right) \\ &= -t\varphi_n^*(p_{k^*})\xi + \left(\varphi_n^*(p_{k^*}) - t\varphi_n^{*'}(p_{k^*})\right)\xi^2 \\ &\qquad + \left(\varphi_n^{*'}(p_{k^*}) - \frac{t}{2}\varphi_n^{*''}(p_{k^*})\right)\xi^3 + o(\xi^3) \\ &= -C_1^2\varphi_n^*(p_{k^*})t \pm \left(2C_1C_2\varphi_n^*(p_{k^*}) - C_1\varphi_n^*(p_{k^*}) - C_1^3\varphi_n^{*'}(p_{k^*})\right)\imath\, t\sqrt{t} \\ &\qquad + o(t\sqrt{t}) \end{aligned}$$

for $t>0$. Identifying the coefficients of t and $t\sqrt{t}$, we obtain

(28)
$$(-1)^{k^*} = -C_1^2\varphi_n^*(p_{k^*}) \quad \text{and} \quad 0 = 2C_1C_2\varphi_n^*(p_{k^*}) - C_1\varphi_n^*(p_{k^*}) - C_1^3\varphi_n^{*'}(p_{k^*}).$$

From the proof of Theorem 2 applied to $\varphi_{n-1} = \varphi_n^*$, we know that $\varphi_n^*(p_{k^*})$ has the same sign as $\prod_{\substack{k=0 \\ k\neq k^*,k*+1}}^{n} (p_{k^*} - p_k)$, i.e., $(-1)^{k^*}\varphi_n^*(p_{k^*}) < 0$ (because of the interlacing property 2). The equations (28) finally yield the constants C_1 and C_2. $\square$

From (25) we obtain, as $t \longrightarrow 0$,

$$h \simeq \sqrt{t}\sqrt{\frac{(p_{k^*+2} - p_{k^*})(p_{k^*} - p_{k^*-1})}{p_{k^*+2} - p_{k^*-1}}},$$

so that $h = O(\sqrt{t})$, i.e., h has the same order as the imaginary part of the root z_0 in (27).

Also, since $h \leq |\Im m(z_0)|$ from the results of Section 4, we must have

$$|\varphi_n^*(p_{k^*})| \leq \frac{p_{k^*+2} - p_{k^*-1}}{(p_{k^*+2} - p_{k^*})(p_{k^*} - p_{k^*-1})} = \frac{1}{p_{k^*} - p_{k^*-1}} - \frac{1}{p_{k^*} - p_{k^*+2}},$$

which, by (12), is satisfied.

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