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Iwasawa L-functions and the Mysterious $\mathcal{L}$-Invariant

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In this talk, we will discuss the appearance of the mysterious $\mathcal{L}$-invariant in Iwasawa L-functions associated to elliptic curves. We will sketch the construction of the Iwasawa L-function, how to determine its first non-vanishing coefficient up to a p-adic unit, and in particular, where the "$\log_p q$" term comes from.

1. Notation

Let p be an odd prime, K a number field, and $\mathcal{O}_K$ its ring of integers. For any place v of K, the completion is denoted by K_v. We normalize the valuation ord_v on K_v so that the image is $\mathbf{Z}$.

Most of what follows works equally well for abelian varieties, but it is technically a little easier to work with elliptic curves. So, let A be an elliptic curve over K. We will abuse the notation somewhat and also use A to denote the Néron model of A over $\mathcal{O}_K$.

Let A^0 be the "connected component" of A containing the identity. We then have a short exact sequence

$$0 \to A^0 \to A \to \Phi \to 0$$

where Φ is the group of connected components of A.

If v is a place of $\mathcal{O}_K$ such that A has split multiplicative reduction at v, then let q_v be a Tate-q parameter for A. That is, one has an exact sequence

$$0 \to q_v^{\mathbf{Z}} \to K_v^* \to A(K_v) \to 0.$$

Let L be a Γ-extension of K. That is, L is Galois over K with $\mathrm{Gal}(L/K) = \Gamma \cong \mathbf{Z}_p$, the additive group of p-adic integers. The isomorphism is not canonical,

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so we fix one by choosing a topological generator γ of Γ. For each K, there is always at least one such extension, namely the cyclotomic Γ-extension, contained in $K(\mu_{p^\infty})$ (K adjoin all p-power roots of unity).

The unique closed subgroup of Γ of index p^n is denoted by Γ_n. Its fixed field is denoted by K_n.

We denote the set of primes of K ramifying in L/K by T. The set of places of K_n lying above T is denoted by T_n and the set of places of L lying above T is denoted by T_L. The subset of T where A has split multiplicative reduction will be denoted by T^{sm}, and the subset where A has good ordinary reduction will be denoted by T^{go}. The order of T^{sm} is denoted by e (this is the expected number of *extra* zeros in the Iwasawa L-function for A and L/K).

We fix a continuous character $\kappa\colon \Gamma \to \mathbf{Z}_p^*$. Then, we define a global p-adic logarithm associated to L/K, $\log_p\colon \mathbb{A}_K^*/K^* \to \mathbf{Z}_p$, as the composition of the maps

$$\mathbb{A}_K^*/K^* \xrightarrow{rec.} \Gamma \xrightarrow{\kappa} \mathbf{Z}_p^* \xrightarrow{\log} \mathbf{Z}_p.$$

Here, $\mathbb{A}_K^*/K^*$ is the idele class group of K, *rec.* is the reciprocity map, and log is Iwasawa's p-adic logarithm.

We then get local p-adic logarithms on the completions K_v^* of K^* by composing with the natural map $K_v^* \to \mathbb{A}_K^*/K^*$.

Let $L_v = LK_v$. If v splits finitely in L, L_v is a Γ-extension of K_v whose Galois group, Γ_v, is a subgroup of Γ.

The character and the p-adic log then satisfy the following compatibility condition: For each $v \in T^{sm}$, the following diagram is commutative

$$\begin{tikzcd} K_v^* \arrow[r, "rec."] \arrow[d, "\log_p"] & \Gamma_v \arrow[d, "\kappa|_{\Gamma_v}"] \\ \mathbf{Z}_p & \mathbf{Z}_p^* \arrow[l, "\log"'] \end{tikzcd} \tag{1}$$

We define the Iwasawa algebra Λ to be the completed group ring $\varprojlim \mathbf{Z}_p[\Gamma/\Gamma_n]$. There is an isomorphism $\Lambda \to \mathbf{Z}_p[[T]]$ induced by sending γ to $T+1$.

For any abelian group, commutative group scheme, or sheaf of abelian groups H, let H_n be the kernel of multiplication by n on H, and $H_{p^\infty} = \varinjlim H_{p^\nu}$.

Finally, for an abelian group H, we let $H^* = \mathrm{Hom}(H, \mathbf{Q}/\mathbf{Z})$, the Pontryagin dual of H. If H is a Λ-module, H^* is also a Λ-module with the action $(\sigma f)(h) = f(\sigma^{-1}h)$ for $f \in H^*$ and $\sigma \in \Gamma$.

2. Assumptions

We place the following assumptions on A, L, and K.

- **A:** the reduction of A at every prime in T is ordinary (i.e., good ordinary or multiplicative);
- **B:** for each place $t \in T^{sm}$, $\log_p(q_t) \neq 0$;
- **C:** the places of K where A has bad reduction split finitely in L;

D: The reduction of A at every prime of K dividing p is semi-stable;
E: the p-primary component of the Tate-Shafarevich group $Ш(K)_{p^\infty}$ is finite;
F: $A(L)_{p^\infty}$ is finite.

These assumptions can be considerably weakened, at the expense of a less aesthetic exposition and statement of results.

3. Defining the Iwasawa L-function

A group homomorphism $f\colon H \to G$ is a quasi-isomorphism from H to G if f has finite kernel and cokernel.

In order to define an Iwasawa L-function, one must produce a compact finitely generated Λ-module M. By a standard classification theorem, M is quasi-isomorphic to a finite direct sum of the form

$$\Lambda^\rho \oplus \bigoplus_i \Lambda/f_i(T) \oplus \bigoplus_j \Lambda/p^{\mu_j}. \tag{2}$$

The $f_i(T)$ can be taken to be distinguished polynomials, that is, monic polynomials congruent to T^{n_i} modulo p where n_i is the degree of f_i. If $\rho = 0$, M is a torsion Λ-module, and its characteristic polynomial

$$F_M(T) := \prod_i f_i(T) \times \prod_j p^{\mu_j}.$$

If M is not torsion, we set $F_M(T) := 0$. For a discrete Λ-module M with compact finitely generated dual M^*, we write $F_M(T)$ for $F_{M^*}(T)$.

The Iwasawa L-function of M is then gotten from F by a change of variables. For an elliptic curve, an appropriate choice is $L(M;s) = F(\kappa(\gamma)^{1-s} - 1)$. So, the order of vanishing of $L(M;s)$ at $s = 1$ is the same as the order of vanishing of $F(T)$ at $T = 0$, and the respective first non-vanishing coefficients differ by a factor of $(-\log_p(\kappa(\gamma)))^\eta$ where η is the common order of vanishing.

In the following, cohomology groups are flat groups for the *fpqf* topology. Everything we need to know about this cohomology is contained in its standard properties (short exact sequences of sheaves giving rise to long exact sequences in cohomology, etc).

There are several natural choices for our analogue of the Selmer group. Here are three:

- $G_1 = H^1(\mathcal{O}_L, A_{p^\infty})$
- $G_2 = H^1(\mathcal{O}_L, A^0_{p^\infty})$
- $G_3 = \text{Image } H^1(\mathcal{O}_L, A^0_{p^\infty}) \to H^1(\mathcal{O}_L, A_{p^\infty})$

It is known that $F_{G_1}(T) = F_{G_2}(T)$ although G_1 and G_2 are in general not quasi-isomorphic. The group G_3 is isomorphic to the classical p-power Selmer group, and $F_{G_3}(T) \cdot T^e = F_{G_1}(T)$. So, it is not difficult to translate information about one L-function to another.

In practice, we found a fourth group to be the easiest to work with, namely,

$$\mathcal{S}_L = \text{Image } H^1(\mathcal{O}_L, A^0_{p^\infty}) \to H^1(\mathcal{O}_L - T_L, A^0_{p^\infty}).$$

This group is quasi-isomorphic to G_3, so $F_{\mathcal{S}_L}(T) = F_{G_3}(T)$. It is convenient to work with since it readily fits into the relative cohomology sequence for $\mathcal{O}_L$ and T_L. We denote the analogous group over K by $\mathcal{S}_K$.

4. The log q term

The residue field of $\mathcal{O}_K$ at v is denoted by k_v. We let r be the rank of $A(K)$. Let $<,>$ denote Schneider's analytic height (see [**7**]), or equivalently, Mazur-Tate canonical height (see [**4**]). Let m_v be the number of connected components for the fiber of A over v. We write simply G for the order of a group G. Finally, $a \approx b$ denotes that a and b differ multiplicatively by a p-adic unit. The basic result is the following.

THEOREM 1.

$$\operatorname{ord}_{s=1} L_p(H^1(\mathcal{O}_L, A^0_{p^\infty}); s) \geq r + e$$

and

$$\frac{1}{(r+e)!}\frac{d^{r+e}}{ds^{r+e}} L_p(H^1(\mathcal{O}_L, A^0_{p^\infty}); s) \approx \left(\prod_{v \in T^{sm}} \frac{\log_p(q_v)}{\operatorname{ord}_v(q_v)}\right)\left(\prod_{v \in T^{go}} A(k_v)^2\right) \frac{\det <,> \text{Ш}(K)_{p^\infty}}{A(K)^2_{p^\infty}} \prod_{v \not| \infty} m_v.$$

It is interesting to consider what happens if the height pairing $<,>$ degenerates. The right side of the above formula is clearly zero. The left side also vanishes. In fact, one can show that if $<,>$ is degenerate, then $\mathcal{S}_L$ is either not a torsion Λ-module (in which case the left side of the formula is identically 0), or $\mathcal{S}_L$ fails a semi-simplicity condition (see below) which accounts for the extra vanishing.

Analogously, one might consider what happens if $\log_p(q_v) = 0$ for a ramifying split multiplicative prime v. In [**5**], Mazur, Tate, and Teitelbaum define what they call the extended Mordell-Weil group of A. The p-adic height pairing on the extended Mordell-Weil group vanishes if and only if a usual p-adic height pairing vanishes or if $\log_p(q_v) = 0$. From this perspective, the behavior when $\log_p(q_v) = 0$ should mirror what happens when $<,>$ degenerates. While one expects that the p-adic L-function would vanish to higher order by at least the number of ramifying split multiplicative places where $\log(q_t)$, only partial results are known at present (to the author). In particular, there is additional vanishing if the elliptic curve has rank 0 over K. One can also show that if A has positive rank and additional zeros do occur, then either $\mathcal{S}_L$ is not torsion, or the additional zeros could not occur semi-simply, just as in the situation where $<,>$ degenerates.

The proof of Theorem 1 is largely contained in [**1**]. Here we will sketch the process for determining the first non-vanishing coefficient of the Iwasawa L-function up to a p-adic unit in the case where $<,>$ is non-degenerate. The Λ-module we will use for the calculation is $\mathcal{S}_L$.

In general, to get at the characteristic polynomial for a finitely generated Λ-module M, we consider the map

$$H^0(\Gamma, M) = M^\Gamma \xrightarrow{g} M_\Gamma = H^1(\Gamma, M).$$

Here M_Γ denotes the co-invariants for Γ, i.e., $M/(\gamma - 1)M$. The map g is then gotten by composing the inclusion $M^\Gamma \to M$ with the projection to the co-invariants.

This map is a quasi-isomorphism if M is a torsion Λ-module and if in its Λ-module decomposition, there are no terms of the form $\Lambda/T^n\Lambda$ with $n > 1$. The second condition is the semi-simplicity condition referred to above and is a potential difficulty. There are naturally occurring torsion Λ-modules which fail the semi-simplicity condition (such as G_2 above). For such modules, some sort of adjustment is necessary since the p-adic height described below will degenerate in that case.

Provided the map g is a quasi-isomorphism, the first non-vanishing coefficient is equal (up to a p-adic unit) to $\operatorname{cok} g/\ker g$. To prove this fact, note that this ratio is invariant under quasi-isomorphism. So, one may assume that the Λ-module has the standard form of equation 2. Then, one is reduced to an elementary computation with power series.

Predominating the calculation of $\operatorname{cok} g/\ker g$ is the p-adic height pairing. It is shown in [**2**] that Schneider's height $<,>$ is equal to $-\log_p(\kappa(\gamma))\cdot <,>_\gamma$ where $<,>_\gamma$ is defined by the following diagram:

$$\begin{tikzcd}
\tilde{A}^0(\mathcal{O}_K) \otimes \mathbf{Z}_p \arrow[d] & \operatorname{Hom}_{\mathbf{Z}_p}(A^0(\mathcal{O}_K) \otimes \mathbf{Z}_p, \mathbf{Z}_p) \\
H^2(\mathcal{O}_K, A_{p^\infty})^* \arrow[d] & (H^1(\mathcal{O}_K, A^0_{p^\infty})^*)_{\mathrm{TOR}} \arrow[u] \\
H^2(\mathcal{O}_K, A^0_{p^\infty})^* & H^1(\mathcal{O}_K, A^0_{p^\infty})^* \arrow[u] \\
{[H^2(\mathcal{O}_L/\mathcal{O}_K, A^0_{p^\infty})/H^1_T(\mathcal{O}_L/\mathcal{O}_K)]^*} \arrow[u] \arrow[d, "\mu^*"] & \mathcal{S}_K^* \arrow[u, "\delta^*"'] \\
{[H^1(\mathcal{O}_L, A^0_{p^\infty})_\Gamma/H^1_T(\mathcal{O}_L)]^*} \arrow[d, equal] & \\
H^1(\Gamma, \mathcal{S}_L)^* \arrow[r, "g^*"] & H^0(\Gamma, \mathcal{S}_L)^* \arrow[uu, "\alpha^*"']
\end{tikzcd}$$

Definitions of the groups in this diagram and the maps are given in [**2**]. The determinant of $<,>_\gamma$ can be calculated by counting the kernels and cokernels of the maps involved. Note that $\ker g$ and $\operatorname{cok} g$ will be involved so that we get an expression for the determinant of the height pairing in terms of the first non-vanishing coefficient of the Iwasawa L-function, and many other terms.

If the pairing $<,>$ is non-degenerate, $<,>_\gamma$ can be simplified to the following diagram (see [**1**]).

$$\begin{tikzcd}
\tilde{A}^0(\mathcal{O}_K)\otimes \mathbf{Z}_p \arrow[d] & \mathrm{Hom}_{\mathbf{Z}_p}(A^0(\mathcal{O}_K)\otimes\mathbf{Z}_p,\mathbf{Z}_p) \\
H^1(\mathcal{O}_K, T_p(\tilde{A})) \arrow[d, equal] & (H^1(\mathcal{O}_K, A^0_{p^\infty})^*)_{\mathrm{Tor}} \arrow[u] \\
H^2(\mathcal{O}_K, A_{p^\infty})^* \arrow[d] & H^1(\mathcal{O}_K, A^0_{p^\infty})^* \arrow[u] \\
H^2(\mathcal{O}_K, A^0_{p^\infty})^* & \mathcal{S}_K^* \arrow[u, "\delta^*"'] \\
H^1(\Gamma,\mathcal{S}_L)^* \arrow[u, "\beta^*"'] \arrow[r, "g^*"] & H^0(\Gamma,\mathcal{S}_L)^* \arrow[u, "\alpha^*"']
\end{tikzcd}$$

Note that the map δ is surjective and the order of its kernel is

$$\frac{A^0_{p^\infty}(\mathcal{O}_K)}{A^0_{p^\infty}(\mathcal{O}_K-T)}\prod_{t\in T} H^1_!(\mathcal{O}_{K_t}, A^0_{p^\infty}).$$

The terms in the fraction are orders of finite groups which enter the calculation elsewhere. The group $H^1_!(\mathcal{O}_{K_t}, A^0_{p^\infty}) = \Phi(\mathcal{O}_{K_t})_{p^\infty}$ and has the same order as the p-primary subgroup of the group of connected components of A at t. This is trivial unless $t \in T^{sm}$, (using that p is odd) in which case it is equal to the p part of $\mathrm{ord}_t\, q_t$, by the Tate parameterization of A. This is the source of the $\mathrm{ord}_t\, q_t$ part of the $\mathcal{L}$-invariant.

For the $\log_p q$ terms, we concentrate on the maps α and β. They are contained in the following exact commutative diagram.

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & \\
& \mathcal{S}_K \arrow[r, "\alpha"] \arrow[d] & \mathcal{S}_L^\Gamma \arrow[d] & \\
A^0_{p^\infty}(\mathcal{O}_L-T)_\Gamma \arrow[r, hook] & H^1(\mathcal{O}_K-T,A^0_{p^\infty}) \arrow[r] \arrow[d] & H^1(\mathcal{O}_L-T,A^0_{p^\infty})^\Gamma \arrow[r] \arrow[d] & 0 \\
E_T \arrow[r, hook] & H^2_T(\mathcal{O}_K,A^0_{p^\infty}) \arrow[r, "\eta"] \arrow[d] & H^2_T(\mathcal{O}_L,A^0_{p^\infty})^\Gamma \arrow[r] \arrow[d] & 0 \\
& H^2(\mathcal{O}_K,A^0_{p^\infty}) \arrow[r, "\beta"] \arrow[d] & H^1(\Gamma,\mathcal{S}_L) \arrow[d] & \\
& H^2(\mathcal{O}_K-T,A^0_{p^\infty}) \arrow[r, equal] \arrow[d] & H^1(\Gamma,H^1(\mathcal{O}_L-T,A^0_{p^\infty})) \arrow[d] & \\
& 0 & 0 &
\end{tikzcd}$$

Just a few words on the source for this diagram (details are given in [**1**]). The main vertical columns come from the relative cohomology sequences over K and L. Over L, the usual long exact sequence terminates as

$$0 \to \mathcal{S}_L \to H^1(\mathcal{O}_L - T, A^0_{p^\infty}) \to H^2_T(\mathcal{O}_L, A^0_{p^\infty}) \to 0 \tag{3}$$

since the hypothesis that the height pairing is non-degenerate combined with the fact that $A_{p^\infty}(L)$ is finite implies that $H^2(\mathcal{O}_L, A^0_{p^\infty}) = 0$. The right column then comes by taking cohomology of Γ for the short exact sequence 3.

By a diagram chase, we have that cok β is trivial and

$$(\ker\beta)\cdot(\text{cok }\alpha)/\ker\alpha = E_T/A^0_{p^\infty}(\mathcal{O}_L - T)_\Gamma.$$

The term we are most concerned with here is the group E_T, which is simply defined as the kernel of the map η. It can be computed as follows.

The group $H^2_{T_K}(\mathcal{O}_K, A^0_{p^\infty}) = \bigoplus_{t\in T_K} H^2_t(\mathcal{O}_K, A^0_{p^\infty})$. Then, the Kummer sequence gives us the short exact sequence

$$0 \to H^1_t(\mathcal{O}_K, A^0)\otimes \mathbf{Q}_p/\mathbf{Z}_p \to H^2_t(\mathcal{O}_K, A^0_{p^\infty}) \to H^2_t(\mathcal{O}_K, A^0)_{p^\infty} \to 0.$$

The first term vanishes since it is the tensor product of a torsion group with a divisible group. So, $H^2_t(\mathcal{O}_K, A^0_{p^\infty}) = H^2_t(\mathcal{O}_K, A^0)_{p^\infty}$. Now, by McCallum's local duality theorem, we have that $H^2_t(\mathcal{O}_K, A^0) = A(K_t)^*$.

So, we have

$$E_T = \bigoplus_{t\in T}(\varprojlim \text{ cok } A(K_{n,t}) \to A(K_t))^*.$$

The map here is the norm map for the elliptic curve. The resulting quotient group in the limit is the quotient of $A(K_t)$ by the subgroup of *universal* norms for the Γ-extension L_t/K_t. Note that if A has good ordinary reduction, this quotient is finite of order $A(k_v)^2$ by the work of Mazur and Schneider [**3, 8**]. If A has non-split multiplicative reduction, the quotient is trivial [**6**].

We now sketch how to calculate this quotient in the case where A has split multiplicative reduction at the place t of K. The result is originally due to Nasybullin [**6**] and can be obtained by using the Tate parameterization.

We start with the diagram

$$\begin{array}{ccccccccc}
0 & \to & q_t^{\mathbf{Z}} & \to & K^*_{n,t} & \to & A(K_{n,t}) & \to & 0\\
 & & \downarrow{\mathbf{N}} & & \downarrow{\mathbf{N}} & & \downarrow{\mathbf{N}} & & \\
0 & \to & q_t^{\mathbf{Z}} & \to & K^*_t & \to & A(K_t) & \to & 0
\end{array}.$$

The downward arrows are norm maps. The exact sequence from the cokernels is then

$$\mathbf{Z}/p^n\mathbf{Z} \to K^*_t/\mathbf{N}(K^*_{n,t}) \to A(K_t)/\mathbf{N}A(K_{n,t}) \to 0.$$

The middle term can be identified by local class field theory with $\mathrm{Gal}(K_{n,t}/K_t)$. Taking the limit up the Γ-tower, this group becomes simply Γ. The cokernel of the local norm mapping on the elliptic curve is then Γ/I where I is the image coming from the cokernels of the first column. These are cyclic (pro-cyclic in the limit) and are generated by the image of q_t. So,

$$\varprojlim A(K_t)/\mathbf{N}A(K_{n,t}) = \Gamma/\overline{<(q_t; L_t/K_t)>}$$

where $(q_t; L_t/K_t)$ is the norm residue symbol of q_t for the extension L_t/K_t. We write Q_t for the image of $(q_t; L_t/K_t)$ under the isomorphism $\Gamma \to \mathbf{Z}_p$ induced by sending γ to 1. Note that although Q_t is not canonically defined (it depends on the choice of the topological generator γ of Γ), a different choice would not change the p-adic valuation of Q_t. Thus, up to p-adic unit $\varprojlim A(K_t)/\mathbf{N}A(K_{n,t}) \approx Q_t$. Using the compatibility condition 1, we find $Q_t \approx \log_p(q_t)/\log_p(\kappa(\gamma))$.

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