OCTIC 2-ADIC FIELDS

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Abstract. We compute all octic extensions of $\mathbf{Q}_2$ and find that there are 1823 of them up to isomorphism. We compute the associated Galois group of each field, slopes measuring wild ramification, and other quantities. We present summarizing tables here with complete information available at our online database of local fields.

1. Introduction

This paper is one of three papers accompanying our online interactive database of low degree p-adic fields, located at `http://math.la.asu.edu/~jj/localfields`. The first paper, [8], describes the database in general and explains why low degree p-adic fields warrant a detailed investigation. The next paper, [9], treats the second hardest case in the range of the database, finding 795 isomorphism classes of nonic 3-adic fields and determining associated invariants.

In this paper, in a parallel way to [9], we treat the hardest case in the range of the database, octic 2-adic fields. We find 1823 isomorphism classes of such fields and determine their associated invariants. The main previous work on octic 2-adic fields is [1] and its sequel [16]. These papers focus on the 130 non-trivial isomorphism classes of Galois extensions of $\mathbf{Q}_2$ with Galois group embeddable into the octic group $\tilde{S}_4 \cong GL_2(3)$.

Sections 2 and 3 catalog basic facts about resolvents and Galois groups respectively. They take place over an arbitrary ground field F of characteristic different from 2. Sections 4 and 5 summarize our findings on octic 2-adic fields. They provide an overview of the actual table of octic 2-adic fields on the database, which would run to approximately forty-five printed pages. These sections also discuss the relation of this paper with [1] and [16] somewhat further, and indicate by reference some applications to number fields.

2. Resolvent Polynomials

Resolvents are an important tool in Galois theory useful for both computing the Galois group of a separable polynomial $f(x) \in F[x]$ and for computing subfields of the splitting field of $f(x)$. In this section we define various resolvents which will be employed for these purposes in Sections 3 and 4, and also to compute slopes in Section 5.

We briefly recall the general construction. If $f(x)$ is a separable degree n polynomial with a chosen ordering of roots $\gamma_1, \ldots, \gamma_n \in \overline{F}$, a separable closure of F, then $G = \mathrm{Gal}(f) \leq S_n$ in the usual way with permutations acting on the γ_i through their subscripts. A resolvent is associated to a pair of subgroups of S_n, $H \leq \Gamma$ where $G \leq \Gamma$.

2000 *Mathematics Subject Classification.* 11S15, 11S20.

Elements $\sigma \in S_n$ give ring automorphisms of $F[x_1, \ldots, x_n]$ induced by $\sigma(x_i) = x_{\sigma(i)}$. From the two groups H and Γ, one can construct a form $A(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n]$ such that the isotropy subgroup of A in Γ is exactly H. Note that the group actions on $\{\gamma_1, \ldots, \gamma_n\}$ and $\{x_1, \ldots, x_n\}$ are chosen so that for $\sigma \in G \leq \Gamma$, they commute with the evaluation map $x_i \mapsto \gamma_i$. The resolvent is then a product over coset representatives:

$$\prod_{\sigma \in \Gamma \backslash H} x - A^{\sigma}(\gamma_1, \ldots, \gamma_n) = \prod_{\sigma \in \Gamma \backslash H} x - A(\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(n)})\,.$$

We refer the resolvent as *absolute* if $\Gamma = S_n$ and otherwise as *relative*. Note that in the case of absolute resolvents, one does not have to take care of ordering the roots of $f(x)$ since $G \leq \Gamma = S_n$ is then automatic.

In §2.1, we define three absolute resolvent constructions, indexed by degree, $f \mapsto f_{28}$, $f \mapsto f_{30}$, and $f \mapsto f_{35}$. More important for the sequel are two relative resolvent constructions, $f \mapsto f_6$ and $f \mapsto f_8$, defined in §2.2 and §2.3 respectively.

Like all resolvent constructions, the ones we present here are defined initially at the level of polynomials, and then automatically pass to the level of fields and separable algebras. Suppose $f(x)$ is separable with $K = F[x]/f(x)$ the corresponding separable algebra. If $f_d(x)$ is separable, then K_d is by definition $F[x]/f_d(x)$. If $f_d(x)$ is not separable, then one presents K differently as $F[x]/g(x)$ such that $g_d(x)$ is separable and defines $K_d = F[x]/g_d(x)$. This process defines K_d up to unique isomorphism. In our applications, typically f is irreducible while the separable polynomial defining K_d may not be. Thus typically K is a field while K_d factors into fields according to how the polynomial defining it factors into irreducibles.

2.1. **Absolute resolvents.** Absolute resolvents of degree m for octic polynomials correspond to conjugacy classes of subgroups in S_8 with index m. The smallest index subgroup is $A_8 = T49$, which has index 2 and corresponds to the usual discriminant. The next two subgroups in terms of index are S_7 and A_7, and their resolvents do not provide new information. Next are

$$[S_8 : S_6 \times S_2] = 28, \qquad [S_8 : T48] = 30, \qquad [S_8 : T47] = 35,$$

with the T-numbering first introduced in [2] and reviewed in the next section. We present the resolvents corresponding to these three subgroups here.

A standard resolvent for $S_6 \times S_2$ is given by

$$f_{\mathrm{disc}}(x) = f_{28}(x) = \prod_{i<j} (x - (\gamma_i - \gamma_j)^2). \tag{2.1}$$

This particular resolvent can be computed algebraically by the formula $f_{28}(x^2) = \mathrm{Resultant}_y(f(y), f(x+y))/x^8$.

For the degree 30 resolvent corresponding to the subgroup $T48 = 2^3 : GL_3(2)$, we use the following from [3]. One starts with the form $A(x_1, \ldots, x_8) = x_1 x_2 x_3 x_4$, and takes its trace with respect to elements of $T48$ acting on subscripts:

$$A_{30}(x_1, \ldots, x_8) = \sum_{\sigma \in T48} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}\,.$$

The result has exactly $T48$ as its isotropy subgroup. Then

$$f_{30}(x) = \prod_{\sigma \in T48 \backslash S_8} (x - A_{30}^{\sigma}(\gamma_1, \ldots, \gamma_8))\,.$$

For the degree 35 resolvent corresponding to $T47 = S_4^2:2$, we follow [13]. Let

$$A_{35}(x_1, \ldots, x_8) = (x_1 + x_2 + x_3 + x_4 - x_5 - x_6 - x_7 - x_8)^2 .$$

Then

$$f_{35}(x) = \prod_{\sigma \in T47 \backslash S_8} (x - A_{35}^{\sigma}(\gamma_1, \ldots, \gamma_8)) .$$

Algebraic expressions for $f \mapsto f_{30}$ and $f \mapsto f_{35}$ are much more complicated than the ones we give for $f \mapsto f_{28}$ above and the relative resolvents below.

2.2. **The sextic relative resolvent.** The sextic resolvent of this subsection is relative in the sense that it requires K to have a quadratic subfield K_2 and moreover depends on the choice of quadratic subfield. Our resolvent corresponds to $\Gamma = T47 = S_4^2:2$ and $H = D_4 \times S_4$, but it is easier to construct as follows. Factoring f over K_2 gives

$$f(x) \quad = \quad (x^4 + ax^3 + bx^2 + cx + d)(x^4 + \bar{a}x^3 + \bar{b}x^2 + \bar{c}x + \bar{d}) .$$

Replacing x by $x - a/4$ on the first factor and by $x - \bar{a}/4$ on the second factor gives a similar expression with coefficients $(1, 0, B, C, D)$ and $(1, 0, \bar{B}, \bar{C}, \bar{D})$. Applying the usual cubic resolvent to each quartic factor gives

$$\begin{aligned} f_{6r}(x) &= \big(x^3 - (3B^2 + 36D)x + (2B^3 + 27C^2 - 72BD)\big) \cdot \\ &\quad \big(x^3 - (3\bar{B}^2 + 36\bar{D})x + (2\bar{B}^3 + 27\bar{C}^2 - 72\bar{B}\bar{D})\big) \\ &= (x^3 + ux + v)(x^3 + \bar{u}x + \bar{v}). \end{aligned}$$

Already $f_{6r}(x)$ is a usable sextic resolvent in $F[x]$. However it is better to apply the mini-twinning operator from Proposition 1 of [17] to get

$$f_6(x) = x^6 - 6u\bar{u}x^4 - 27v\bar{v}x^3 + 9u^2\bar{u}^2x^2 + 81u\bar{u}v\bar{v}x - (4u^3\bar{u}^3 + 27\bar{u}^3v^2 + 27u^3\bar{v}^2).$$

One advantage of f_6 over f_{6r} is that one gets the clean statement (3.4). Another advantage is that a column K_{6r} in Table 3.3 or Table 3.5 would have all entries irreducible, while the the entries in the given K_6 column are mostly reducible, making it easier to distinguish octic Galois groups.

The only computationally non-trivial part of the construction $f \mapsto f_6$ is the initial factorization of f over its quadratic subfield. This can be reduced to factoring a degree 16 polynomial over F by Algorithm 3.6.4 of [4]. It is thus substantially simpler than factoring the absolute resolvents of the previous subsection.

2.3. **The octic relative resolvent.** The octic resolvent of this subsection is relative in the sense that it requires the field K to have a quartic subfield K_4 and moreover depends on the choice of quartic subfield. Here $\Gamma = T44 = 2^4:S_4$ and $H = S_4 \times C_2$, but it is again easier to work directly with polynomials. If

$$f(x) = x^8 + ax^6 + bx^4 + cx^2 + d$$

with K_4 defined by $f_4(x) = x^4 + ax^3 + bx^2 + cx + d$ then we can choose the γ_i such that $\gamma_{4+i} = -\gamma_i$ for $i = 1, 2, 3, 4$. Then our resolvent, the product being over the eight possible sign combinations, is

$$\begin{aligned} f_8(x) &= \prod (x - (\gamma_1 \pm \gamma_2 \pm \gamma_3 \pm \gamma_4)^2) \\ &= x^8 + 8a\, x^7 + (28a^2 - 16b)\, x^6 + (56a^3 - 96ab + 128c)\, x^5 \\ &\quad + (70a^4 - 240a^2b + 512ac + 96b^2 - 2176d)\, x^4 \\ &\quad + (56a^5 - 320a^3b + 768a^2c + 384ab^2 - 2560ad - 1024bc)\, x^3 \\ &\quad + (28a^6 - 240a^4b + 512a^3c + 576a^2b^2 + 1280a^2d - 2048abc - 256b^3 \\ &\qquad - 7168bd + 4096c^2)\, x^2 \\ &\quad + (8a^7 - 96a^5b + 128a^4c + 384a^3b^2 + 1536a^3d - 1024a^2bc - 512ab^3 \\ &\qquad - 6144abd + 2048b^2c + 8192cd)\, x \\ &\quad + a^8 - 16a^6b + 96a^4b^2 - 128a^4d - 256a^2b^3 + 1024a^2bd + 256b^4 \\ &\qquad - 2048b^2d + 4096d^2\,. \end{aligned}$$

This direct formula for f_8 was obtained by first expanding the product in terms of the γ_i. The resulting coefficients are symmetric polynomials in the γ_i which we then expressed them in terms of the elementary symmetric polynomials a, b, c, and d.

A point to note in this construction is that the discriminant of $f(x)$ is d times the square $(16\,\mathrm{disc}(f_4(x)))^2$ while the discriminant of $f_8(x)$ is a perfect square in F. Another point is that $f_8(x)$ factors over $F(\sqrt{d})$.

When applying this construction to a field K, the automorphism σ for K/K_4 may not be given by $\sigma(x) = -x$. Given $\sigma(x) = \sum_{i=0}^{7} a_i x^i$, we can replace f by g, the characteristic polynomial of $x - \sigma(x)$, and then compute g_8 using the formula above.

3. Galois theory of octics

Let $f(x) \in F[x]$ be an irreducible octic polynomial. Let $K = F[x]/f(x)$ be its root field and let $K^g \subset \overline{F}$ be its splitting field. The group $\mathrm{Gal}(K^g/F)$ is a transitive subgroup of S_8, well-defined up to conjugation. We describe in §3.1-3.2 how we determine $\mathrm{Gal}(K^g/F)$ among the fifty such octic groups.

We say that octic fields K and K' are *siblings* if their splitting fields coincide. Note that if K and K' are defined by irreducible polynomials $f(x)$ and $f'(x)$ respectively, it is possible that the T-numbers for $\mathrm{Gal}(f)$ and $\mathrm{Gal}(f')$ are different. In this case, $\mathrm{Gal}(f)$ is isomorphic to $\mathrm{Gal}(f')$ as abstract groups, but the two groups are not conjugate to each other in S_8. In §3.1-3.2 we also discuss ways of constructing siblings of a given field K. In §3.3 we focus on this phenomenon, finding that sibling sets can contain 1, 2, 3, 4, 6, or 8 isomorphism classes of fields, according to the isomorphism class of the common Galois group $\mathrm{Gal}(K^g/F)$.

The standard way ([3],[13]) to determine $\mathrm{Gal}(K^g/F)$ is to factor the high degree resolvents f_{28}, f_{30}, and f_{35}, and occasionally also resolvents of even higher degree. Our procedure is different, as we stay away from high degree resolvents as much as possible, using the sextic and octic relative resolvents of §2.2 and §2.3 instead. For $F = \mathbf{Q}_2$, one can *a priori* rule out ten of the fifty octic Galois groups. We use this along with data for K/F to compute our octic Galois groups working only in fields of degree ≤ 8. The role of the high degree resolvents for us is to provide a quick and easy way of constructing siblings. Note, while simple considerations leave 40 candidates for the Galois group of an octic, only 38 of these groups actually occur.

3.1. **Five types of octic fields.** For ground field $F = \mathbf{Q}_2$, among the easiest invariants to compute of a given field K is the lattice of subfields. Accordingly, our approach throughout this section is to use subfields as much as possible.

We begin by dividing octic fields K into five types according to their intermediate subfields $F \subset K_* \subset K$ as follows. Type 24C consists of fields containing a chain

$$F \subset K_2 \subset K_4 \subset K. \tag{3.1}$$

Type 24P consists of fields having exactly one subfield K_2 of degree two and one subfield K_4 of degree 4, with K_4 not containing K_2. This condition forces

$$K = K_4 \otimes K_2. \tag{3.2}$$

In our notation, "C" stands for chain and "P" for product, in accordance with (3.1) and (3.2) respectively. Type 4 consists of fields with exactly one intermediate subfield, with this subfield having degree 4. Type 2 consists of fields with exactly one intermediate subfield, with this subfield having degree 2. Finally, type Prim consists of fields with no intermediate subfields.

For general ground fields F, the largest possible Galois group G_{gen} for a field of a given type is given in Table 3.1. All groups of a given type are subgroups of G_{gen}

Table 3.1. Generic Galois groups for the five types of octic fields. Also degrees of the factor fields of the corresponding resolvent algebras K_{28}, K_{30}, and K_{35}.

Type	G_{gen}			$\lvert G_{\text{gen}} \rvert$	K_{28}	K_{30}	K_{35}	#
24C	$T35$	P	$2 \wr 2 \wr 2$	2^7	$16\ 8'_{35}\ \mathbf{4}$	$16\ 8_{31}\ 4\ 2$	$16\ 8''_{35}\ 8_{29}\ 2\ \mathbf{1}$	26
24P	$T24$	$S_4 \times 2$		$2^4 3$	$\mathbf{12\ 12\ 4}$	$8\ \mathbf{6\ 6\ 3\ 3\ 2\ 1\ 1}$	$\mathbf{12}\ 12\ \mathbf{4\ 3\ 3\ 1}$	3
4	$T44$	W	$2 \wr S_4$	$2^7 3$	$\mathbf{24\ 4}$	$16\ 12\ 2$	$\mathbf{24}\ 8_{41}\ 3$	7
2	$T47$	$2^4 : 3^2 : D_4$	$S_4 \wr 2$	$2^7 3^2$	$\mathbf{16\ 12}$	$24\ 6_{13}$	$18\ \mathbf{16\ 1}$	7
Prim	$T50$	S_8		$2^8 3^2 5^1 7$	$\mathbf{28}$	30	35	7

and the number of them is given in the last column labeled #. As is standard, m indicates a cyclic group of order m, $\times$ a Cartesian product, and $:$ a semi-direct product. Also $\wr$ denotes a wreath product, as in $S_4 \wr 2 = S_4^2 : 2$, $2 \wr S_4 = 2^4 : S_4$, or $2 \wr 2 \wr 2 = 2^4 : 2^2 : 2$. We systematically use the standard "T-numbering" first introduced in [2]. We also use various more descriptive names. Here P stands for Prime, as P is the Sylow 2-subgroup of S_8. Also W stands for Weyl, as W is the Weyl group of the root lattices B_4 and C_4.

The numbers in the column K_d, ignoring their subscripts and superscripts, give the degrees of the factor fields of K_d when $G = G_{\text{gen}}$. A degree is printed in bold if and only if the corresponding factor remains irreducible for all G within the given type. Thus, for example, the table shows that factoring K_{28} is an alternative way of distinguishing the five types.

The columns K_d also contain the first instances of a convention in force for the five tables in the next subsection as well. This convention is that subscripts on degrees indicate Galois groups via T-numbering while superscripts distinguish isomorphism classes within a row. Thus, a field of type 8_{35} is an octic field whose splitting field has Galois group $T35$. For such a field, the first row says that the high degree resolvents construct all together four more octic fields, $8'_{35}$, 8_{31}, $8''_{35}$, and 8_{29}. In §3.3 we will see that 8_{35}, $8'_{35}$, $8''_{35}$ are three fields from a sibling set of size eight, while 8_{31} and 8_{29} are two fields from a sibling set also of size eight.

The other subscripted entries on Table 3.1 represent factorizations

$$K_{35} = K_{24}K_8K_3, \tag{3.3}$$

$$K_{30} = K_{24}K_6 \tag{3.4}$$

for Type 4 and Type 2 fields respectively. The K_8 on the right of (3.3) is just the octic resolvent of K; this can be seen directly from the fact that if α is a root of $f_8(x)$ then 4α is a root of $f_{35}(x)$. The K_6 on the right of (3.4) is just the sextic resolvent of K. Since both the constructions $K \to K_{30}$ and $K \to K_6$ are complicated, this is best seen by computing one example with $G = S_4 \wr 2$, say $f(x) = (x^4 + \sqrt{2}x + \sqrt{2})(x^4 - \sqrt{2}x - \sqrt{2})$ with ground field $\mathbf{Q}$.

The five groups listed in Table 3.1 will serve as ambient groups for us, meaning that we will consider subgroups inside of them. It works out that

$$\begin{aligned} |G| \text{ has the form } 2^a \quad &\Leftrightarrow \quad G \text{ has type 24C},\\ |G| \text{ has the form } 2^a \cdot 3 \text{ or } 2^a \cdot 3^2, \quad &\Leftrightarrow \quad G \text{ has type 24P, 4, or 2},\\ 7 \text{ divides } |G| \quad &\Leftrightarrow \quad G \text{ has type Prim.} \end{aligned}$$

The fact that our division into five types is so nicely connected with the primes dividing $|G|$ should be viewed as coincidental. For example, $\mathrm{Aff}_2(3) = 3^2 : GL_2(3)$ is a primitive subgroup of S_9 with order of the form 2^a3^b.

3.2. **Distinguishing groups within the five types.** We now consider the possible octic Galois groups by type. Tables 3.2–3.6 show how to distinguish Galois groups within a type using, as much as possible, automorphisms, subfield information, and low degree resolvents.

For each group, we give its T-number in the first column labeled G, descriptive names, its order, and information for distinguishing it from other Galois groups. For example, where useful, we give the automorphism group $\mathrm{Aut}(K/F)$ of the octic extension in the column A, and quartic subfields in a column K_4.

Also Tables 3.2–3.6 give the number of fields with the given T-number in a sibling set in the column labeled m. Finally, in the last column #, the tables give the number of octic 2-adic fields with the given group. If the group can be ruled out for simple reasons as a possible Galois group over $\mathbf{Q}_2$, a hyphen is used in place of zero.

The column labeled "1" in each of the following tables gives discriminant information. The discriminant of the octic field is represented as c. When there is a unique quartic or a unique quadratic subfield, its discriminant is represented by b or a respectively. The 1-column indicates when these values or products of them are squares. Thus, if the group is an even subgroup of S_8, a c will be listed. Similarly, if there is a unique quartic subfield and its Galois group is an even subgroup of S_4, a b will be listed. An entry such as bc indicates that the product of the discriminants of the quartic subfield and the octic field itself is a square. If both b and c are squares, we list b, c instead.

3.2.1. *Type* $24C$ *fields.* Of the 50 conjugacy classes of transitive subgroups of S_8, 26 of them are of type 24C. Most distinctions can be made by considering a combination of subfields, automorphism groups, and discriminant classes. The remaining distinctions can be made with the octic resolvent K_8.

The fields of type $24C$ are presented in Table 3.2. We group them by the number and Galois types of their quartic subfields. In each case, the sextic resolvent f_6 factors as a product of an irreducible quartic times an irreducible quadratic. In

Table 3.2. Galois groups G of fields K of Type 24C. They are distinguished by their quartic subfields, automorphism groups, and octic resolvents K_8. For these groups, K_{28}, K_{30}, and K_{35} have many factors. Only the octic factors with isomorphic Galois group are listed, with 8_i^* being abbreviated i^*. Thus, e.g., the 16 $8_{35}'$ **4** | 16 8_{31} 4 2 | 16 $8_{35}''$ 8_{29} 2 **1** on the first line of Table 3.1 becomes simply $35'$ | | $35''$ here.

G	**Name**	$\lvert G\rvert$	A	K_4's	1	K_6	K_8	K_{28}	K_{30}	K_{35}	m	#
2	C_4C_2	2^3	C_4C_2	$C_4^2V_4$	c							18
3	C_2^3	2^3	C_2^3	V_4^7	c							1
4	D_4^{oct}	2^3	D_4	$V_4D_4^2$	c							18
9	D_4C_2	2^4	V_4	$V_4D_4^2$	c			$9'\ 9''$	$9'''$	$9'\ 9''$	4	36
10	$2^2:4$	2^4	V_4	$C_4D_4^2$	c	$C_2C_1^4$			$10'$		2	24
18	$V\wr 2$	2^5	V_4	D_4^3	c	$C_2C_1^4$			$18'$		8	32
1	C_8	2^3	C_8	C_4	ab,bc	C_4C_2	C_4C_4					24
7	$8:\{1,5\}$	2^4	C_4	C_4	ab,bc	C_4C_2	$T2$					36
16	$\frac{1}{2}[2^4]4$	2^5	C_2	C_4	ab,bc	C_4C_2	$T10$	$16'$		$16'$	2	24
20	$2^3.4$	2^5	C_2	C_4	c	V_4C_2	D_4D_4'	21	$19\ 19'$	21	$1a$	24
27	$2^4.4$	2^6	C_2	C_4	ab	D_4C_2	$T19$	28		28	$2b$	96
5	Q_8	2^3	Q_8	V_4	b,c		V_4V_4					6
11	$Q_8:2$	2^4	V_4	V_4	b,c		V_4V_4'	$11'\ 11''$	$11'\ 11''$	$11'\ 11''$	3	48
21	$\frac{1}{2}2^42^2$	2^5	V_4	V_4	b		$T10$	$19\ 19'\ 20$		$19\ 19'\ 20$	$1a$	24
22	$2^3:D_4$	2^5	C_2	V_4	b,c		V_4V_4'	$22'\ 22''\ 22'''$	$22^4\ 22^5$	$22'\ 22''\ 22'''$	6	–
31	P^b	2^6	C_2	V_4	b		$T18$	$29\ 29'\ 29''$		$29\ 29'\ 29''$	$2c$	32
6	$8:\{1,7\}$	2^4	C_2	D_4	bc	D_4C_2	D_4D_4	$6'$	$6'$	$6'$	2	32
8	$8:\{1,3\}$	2^4	C_4	D_4	bc	D_4C_2	$T4$					36
15	$8:8^\times$	2^5	C_2	D_4	bc	D_4C_2	$T9$	$15'$		$15'$	2	76
17	$4\wr 2$	2^5	C_4	D_4	bc	D_4C_2	$T10$			$17'$	2	96
19	$2^3:4$	2^5	C_2	D_4	c	V_4C_2	D_4C_4	21	$19'\ 20$	21	$2a$	48
26	P^{bc}	2^6	C_2	D_4	bc	D_4C_2	$T18$	$26'$		$26''$	4	96
28	P^{ac}	2^6	C_2	D_4	ac	C_4C_2	$T20$	27		$27'$	$2b$	96
29	P^c	2^6	C_2	D_4	c	V_4C_2	D_4D_4'	31	$29'\ 29''$	$31'$	$6c$	96
30	P^{abc}	2^6	C_2	D_4	abc	D_4C_2	$T19$	$30'$		$30''$	4	32
35	P	2^7	C_2	D_4		D_4C_2	$T29$	$35'$		$35''$	8	384

the case of extensions of $\mathbf{Q}_2$, $T22$ can be ruled out *a priori* because it has C_2^4 as a quotient group.

In column m, the isomorphy classes $\{T19, T20, T21\}$, $\{T27, T28\}$, and $\{T29, T31\}$ are indicated by a, b, and c respectively. Our labeling indicates, for example, that $T31$ octic fields come in pairs sharing a splitting field K^g, and for each pair one has six $T29$ fields with the same splitting field K^g.

3.2.2. *Type* $24P$ *fields.* The three groups corresponding to type 24P fields are given in Table 3.3. Since the quartic subfield does not contain the quadratic subfield, the

TABLE 3.3. Galois groups G of a field K of type 24P. They are distinguished by either their sextic or octic resolvents.

G	**Name**	$\lvert G\rvert$	K_4	1	K_6	K_8	m	#
13	$A_4 \times 2$	$2^3 3$	A_4	b,c	$C_3C_2C_1$	$A_4C_3C_1$		7
14	S_4^{oct}	$2^3 3$	S_4	ab,c	$S_3C_1C_1C_1$	$S_4S_3C_1$		3
24	$S_4 \times 2$	$2^4 3$	S_4	c	$S_3S_2C_1$	$S_4S_3C_1$	2	18

octic field is necessarily the compositum of these two subfields.

3.2.3. *Type* 4 *fields.* The seven possible Galois groups of a Type 4 field are as in Table 3.4. Weil [21] showed that there are no extensions of $\mathbf{Q}_2$ with Galois group

TABLE 3.4. Galois groups G of a field K of type 4. They are distinguished by their octic resolvents K_8. For a type 4 field one has a factorization $K_{30} = K_{16}K_{12}K_2$ and the factorization of K_{16} is indicated.

G	**Names**		$\lvert G\rvert$	K_4	1	K_8	K_{16}	m	#
12	$\tilde{A}_4$	$SL_2(3)$	$2^3 3$	A_4	b,c	A_4A_4	$8_{12}\ 8_{12}$		0
23	$\tilde{S}_4$	$GL_2(3)$	$2^4 3$	S_4	bc	8_{14}	16	2	16
32	$W^{b,c}$	$2^3 : A_4$	$2^5 3$	A_4	b,c	$A_4'A_4''$	$8_{32}'\ 8_{32}''$	3	–
38	W^b	$2^4 : A_4$	$2^6 3$	A_4	b	8_{33}	16	2	56
39	W^c	$2^{3c} : S_4$	$2^6 3$	S_4	c	$S_4'S_4''$	$8_{39}'\ 8_{39}''$	6	0
40	W^{bc}	$2^{3bc} : S_4$	$2^6 3$	S_4	bc	8_{34}	16	2	8
44	W	$2^4 : S_4$	$2^7 3$	S_4		8_{41}	16	4	144

$T12 = SL_2(3) = \tilde{A}_4$. The group $T32$ is not possible because it has 5 quotients isomorphic to A_4, whereas $\mathbf{Q}_2$ has only one A_4 extension. Like in Weil's $T12$ case, the reason for the non-existence of $T39$ fields is subtle, depending on the non-vanishing of an embedding obstruction; like for $T12$, there are extensions of $\mathbf{Q}_2$ with Galois group every proper quotient of $T39$.

3.2.4. *Type* 2 *fields.* The seven possible Galois groups of a Type 2 field are listed in Table 3.5. The groups $T45$, $T46$, and $T47$ are ruled out as possible Galois groups by the resolvent field K_6, since $\mathbf{Q}_2$ does not have sextic extensions with Galois groups $S_3 \times S_3$, $C_3^2 : C_4$, or $C_3^2 : D_4$ (see [8]).

TABLE 3.5. Galois groups G of a field K of type 2. They are distinguished by the sextic resolvent K_6, which factors in the first five cases but not in the last two. For a type 2 field one has the a factorization $K_{30} = K_{24}K_6$ and the factorization of K_{24} is also indicated.

G	**Name**	$\|G\|$	1	K_6	K_{24}	m	#
33	$2^4 : A_3 2$	$2^5 3$	c	$C_3C_2C_1$	$12\ 8'_{33}\ 4$	2	14
34	$2^4 : S_3$	$2^5 3$	c	$S_3C_1C_1C_1$	$12\ 4\ 4'\ 4''$		1
41	$2^4 : S_3 2$	$2^6 3$	c	$S_3C_2C_1$	$12\ 8'_{41}\ 4$	2	36
42	$2^4 : S_3 A_3$	$2^5 3^2$	c	S_3C_3	12 12		5
45	$2^4 : S_3 S_3$	$2^6 3^2$	c	S_3S_3	12 12		–
46	$2^4 : 3^2 : 4$	$2^6 3^2$	ac	$C_3^2 : C_4$	24		–
47	$S_4 \wr 2$	$2^7 3^2$		$C_3^2 : D_4$	24		–

3.2.5. *Primitive fields.* The seven possibilities for the Galois group G of a primitive field K are listed in Table 3.6. The factor partition of K_{30} distinguishes the seven possibilities, except for $T25$, $T36$, and $T48$, all of which have factor partition 14, 8, 7, 1. This ambiguity is removed by considering the Galois group of the degree seven factor, which has size $|G|/8$, thus 7, 21, or 168 in the respective cases. The last five groups on Table 3.6 are non-solvable, and so cannot appear as a Galois group over $\mathbf{Q}_2$.

TABLE 3.6. Galois groups G of a primitive field K. Over general fields, they are distinguished by the factorization partition of K_{30} together with the Galois groups of the septic factor of K_{30}. Over $\mathbf{Q}_2$ only $T25$ and $T36$ arise and $G = T25$ if and only if $\mathrm{ord}_2(\mathrm{disc}(K)) = 14$.

G	**Name**	$\|G\|$	1	K_{30}	K_{35}	m	#
25	$2^3 : 7$	$2^3 7$	c	$14\ 8_{25}\ 7_1\ 1$	$28\ 7_1$		2
36	$2^3 : 7 : 3$	$2^3 3^1 7$	c	$14\ 8_{36}\ 7_3\ 1$	$28\ 7_3$		14
37	$PSL_2(7)$	$2^3 3^1 7$	c	$7_5\ 7_5\ 7'_5\ 7'_5\ 1\ 1$	21 14		–
43	$PGL_2(7)$	$2^4 3^1 7$		14 14 2	21 14		–
48	$\mathrm{Aff}_3(2)$	$2^6 3^1 7$	c	$14\ 8'_{48}\ 7_5\ 1$	$28\ 7'_5$	2	–
49	A_8	$2^6 3^2 5^1 7$	c	15 15	35		–
50	S_8	$2^7 3^2 5^1 7$		30	35		–

The method of inspecting the septic factor K_7 of K_{30} is computationally feasible. In fact, there are only two septic 2-adic fields, the unramified one with Galois group $7_1 = 7$ and the totally ramified one with Galois group $7_3 = 7 : 3$.

However it is even easier to distinguish between $T25$ and $T36$ using a general fact about primitive extensions K of $F = \mathbf{Q}_p$ of degree p^m and discriminant exponent $c = \mathrm{ord}_p(\mathrm{disc}(K))$. In the language of Section 5, the Galois slope content of K

must be $[s, \cdots, s]_t^u$. Here the wild slope s is $c/(p^m - 1)$ and is repeated m times. It satisfies $1 < s \leq 2 + \frac{1}{p-1}$ with equality possible if and only if $m = 1$. The quantity t, measuring tame ramification, is the denominator of s, i.e. $(p^m - 1)/\gcd(c, p^m - 1)$. In our setting $p^m = 8$, one has $t = 1$ if and only if $c = 14$.

3.3. **Siblings.** Let K be an octic field and put $G = \mathrm{Gal}(K^g/F)$ following our standard notations. Consider subgroups H of index 8 such that the intersection of H with its conjugates is the identity group. Let S be the set of conjugacy classes of such groups. Then S naturally indexes the set of siblings of K. The T-number for an element of S corresponds to the action of G on the cosets $H\backslash G$. The size of S and the corresponding T-numbers of its elements can be easily computed with [5].

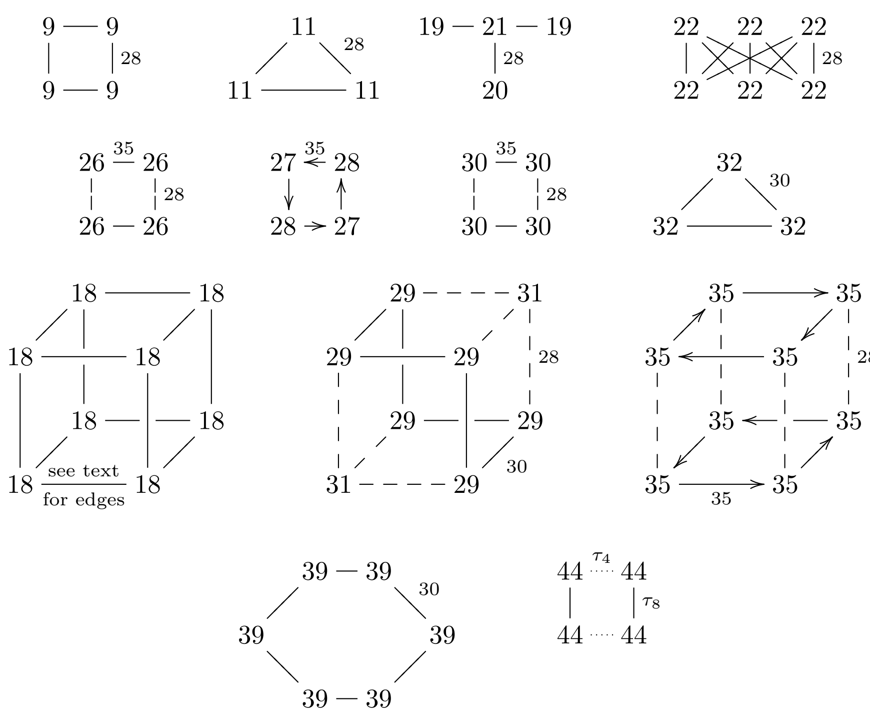

FIGURE 3.1. Octic sibling sets of size ≥ 3. Resolvent constructions for computing siblings are indicated by their degree, and quadratic twists by τ_*.

To explicitly construct the sibling set of an octic field K, the resolvents we have been using do not always suffice. They need to be supplemented by quadratic twisting as follows. Suppose given a quartic subfield K_4 of K and an element $d \in F^\times/F^{\times 2}$. Then if $K_4 = F[x]/g(x)$ and $K = F[x]/g(x^2)$ the associated quadratic twist is $K^d = F[x]/g(dx^2)$. One clearly has $(K^d)^g F(\sqrt{d}) = K^g F(\sqrt{d})$. One has the desired $(K^d)^g = K^g$ if and only if $\sqrt{d}$ is in both $(K^d)^g$ and K^g.

The simplest situation is when the sibling set has size two, i.e. twins. This occurs for exactly the groups Ti with $i \in \{6, 10, 15, 16, 17, 24, 33, 41, 48\}$ or $i \in \{23, 38, 40\}$. For the first nine groups, the twin of a given field K is obtained via resolvents, as specified on Tables 3.2–3.6. For the last three groups, quadratic twisting by the discriminant of K gives the twin. In all twelve cases, it happens that a field and its twin share the same T-number.

The larger sibling sets are described by the graphs in Figure 3.1. The vertices give the T-numbers for the Galois group of the corresponding field. The edges are resolvent or twisting constructions. We use different edge-styles for different constructions and label one edge of each type. We put an arrow on an edge if the construction works in only one direction. If repeating the construction takes us back, then we put no arrows on the edge rather than two arrows. Often resolvent and twisting constructions that are not included on the diagram also connect vertices; we have included only enough to make each graph connected. In fact, we need twisting only for $T44$ and $T18$. On the graph for $T44$, τ_k indicates twisting by means of the discriminant of the degree k subfield.

The case of $T18$ is the most complicated. Let K be an octic $T18$ field. Figure 3.1 indicates K by one vertex of a cube and its seven remaining siblings by the other vertices. In the degree 32 Galois extension K^g/F, there are six octic subfields with Galois group D_4, say $F_1, \ldots, F_6$. These fields correspond to the six faces of the cube. A vertex for a field K is at the corner of the face for F_i if and only if F_i is the splitting field of a quartic subfield of K. Thus, the figure reflects the fact that K has three quartic subfields with different splitting fields.

Given a $T18$ field K, we can pick two quartic subfields K_4 and K_4'. These choices correspond to two faces of the cube which then share an edge e which has one vertex at K. If we twist using K_4 and $\mathrm{disc}(K_4')$, or equally well K_4' and $\mathrm{disc}(K_4)$, we move along the edge e. In contrast, if we twist using K_4 and $\mathrm{disc}(K_4)$, we move diagonally across the face corresponding to the splitting field of K_4; so this simpler type of twisting is not by itself sufficient to pass from K to all its siblings.

4. Octic 2-adic Fields by Discriminant and Galois Group

The main result of this paper is a detailed description of the set of octic 2-adic fields. The full description is at our website and Table 4.1 presents an overview. Even Table 4.1 is somewhat complicated and in this section we describe it part-by-part.

4.1. Mass formula totals.

One of the easier invariants to compute of an octic 2-adic field K is its automorphism group A. The mass of K is then defined to be $1/|A|$. Table 4.1 gives $|A|$, which depends only on the Galois group $G = \mathrm{Gal}(K^g/\mathbf{Q}_2)$. Note, however, that it is much easier to compute A than it is to compute G.

Let $f \in \{1, 2, 4, 8\}$ and put $e = 8/f$. Let $\mathcal{K}_{e,f,c}$ be the set of isomorphism classes of octic 2-adic fields with residual degree f, ramification degree e, and discriminantal ideal (2^c). General p-adic mass formulas, [12, 20, 15], give the total mass of $\mathcal{K}_{e,f,c}$. The set $\mathcal{K}_{1,8,0}$ has one element, the unramified octic extension of $\mathbf{Q}_2$; so its total mass is 1/8. Table 4.1 gives on the lines M_e the mass $M_{e,f,c}$ of the remaining nonempty $\mathcal{K}_{e,f,c}$.

4.2. The main table entries.

Once one knows the mass of $\mathcal{K}_{e,f,c}$, there is a procedure for obtaining a defining polynomial $f_i(x)$ for each element of $\mathcal{K}_{e,f,c}$ [15]. Carrying out this procedure, we find the desired $f_i(x)$. Since the distribution of the $|A|$ was not known beforehand, it is only at this point that we know the numbers $|\mathcal{K}_{e,f,c}|$, not just their lower bounds $M_{e,f,c}$. The numbers $|\mathcal{K}_{e,f,c}|$ are listed on the

TABLE 4.1. The 1823 octic 2-adic fields by discriminant and Galois group

G	$\|A\|$	0	8	10	12	14	16	17	18	20	21	22	24	25	26	27	28	29	30	31	**Tot**	m	2	2,p_3	2,p_5
1	8	***1***	**1**		**2**							*4*								16	24		2	4	8
2	8		**1**		**2**		*3*					*4*	8								18		1	6	18
3	8						*1*														1			1	1
4	8				*2*		*2*					*2*,8	4								18		2	14	12
5	8											*2*	4								6			2	
6	2											*2*	2			8	8			12	32	2	4	20	20
7	4		**1**		**1**		*4*					*6*								24	36		1	6	20
8	2						2		2			*2*	2			8	8			12	36		2	22	10
9	4						*4*		*4*,4	*4*		*4*,8	8								36	4		28	24
10	4		**1**		**1**,*1*		*5*								16						24	2	2	8	24
11	4						*4*		*4*	*4*,8		*8*,4	16								48	3		18	18
15	2						*8*			4		*4*	4			16	16			24	76	2		42	42
16	2		**1**		**1**,*1*		*1*					*4*								16	24	2	2	8	24
17	4						4		4	8			16			32				32	96	2	4	16	72
18	4									8		8			16						32	8		24	8
19	2					2			2	8		4			16		16				48	2a	2	8	24
20	2		**1**		**1**		*2*			*4*							16				24	a	1	4	12
21	2				*1*		*3*		*4*					16							24	a	1	4	12
26	2								*8*	*8*						16	32			32	96	4		64	24
27	2		**4**		**4**				*8*	*8*		*8*							32	32	96	2b	4	16	48
28	2				*4*		*4*		*8*	*8*		*8*						32		32	96	2b	4	16	48
29	2								*8*			*8*	16		32		32				96	6c		48	24
30	2						*8*		*8*	*8*		*8*						32	32		96	4	4	16	48
31	2									*8*		*8*		16							32	2c		16	8
35	2										32		32	32	32	64		64	64	64	384	8		168	72
13	2				*1*	2			4												7				
14	2		*1*				*2*														3				
24	2				2		4		4	8											18	2			
23	2			2	2		4					8									16	2			
38	2						4		4	16	32										56	2			
40	2											8									8	2			
44	2					8	8	32					32	64							144				
33	1				*2*					4			8								14	2			
34	1						*1*														1				
41	1						2		2			16			16						36	2			
42	1		*1*				*4*														5				
25	1					2															2				
36	1		1	1	2		2		4	4											14				
$\#_2$			**10**		**12**																**22**				
M_2			**3.75**		**4**																				
$\#_4$			*2*		*12*		*56*		*52*	*52*		*82*									*256*				
M_4			*1.5*		*6*		*24*		*24*	*24*		*32*													
$\#_8$			1	3	6	14	30	32	30	68	64	64	152	128	128	144	128	128	128	296	1544				
M_8			1	2	4	8	16	16	16	32	32	32	64	64	64	64	64	64	64	128					

lines $\#_e$ of Table 4.1. Summing over all e, f, c, we find that there are 1823 octic 2-adic fields in all. For a general treatment of passing from the simple numbers $M_{e,f,c}$ to the more complicated numbers $|\mathcal{K}_{e,f,c}|$ in the context $\mathrm{ord}_p(e) \leq 2$, see [6].

The actual defining polynomials $f_i(x)$ for the fields are listed at our website. As described in [8], the website gives many invariants of each $K_i = \mathbf{Q}_p[x]/f_i(x)$. Some of the invariants, such as subfields, are used in the computation of Galois groups, as explained in the previous section. Naturally G itself is given on the website too. Table 4.1 gives the number $|\mathcal{K}_{e,f,c,G}|$ of fields with a given (e, f, c, G). Numbers in ordinary, italic, bold, and bold italic type, correspond to $e = 8$, 4, 2, and 1 respectively.

4.3. **Galois group totals.** The column "Tot" gives the total number $|\mathcal{K}_G|$ of octic 2-adic fields with a given octic Galois group $G = Ti$. The column "m", as before, including the a, b, c labeling convention explained after Table 3.2, gives the number m_G of such octic fields having a given Galois field K^g as splitting field. Necessarily, m_G divides $|\mathcal{K}_G|$.

When G is a 2-group, the number $|\mathcal{K}_G|/m_G$ can be obtained in an alternative way as follows. Let $\mathbf{G}$ be the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}_2/\mathbf{Q}_2)$. Let $\mathbf{G}(2)$ be its maximal pro-2 quotient. Then an explicit presentation by generators and relations is known, namely $\mathbf{G}(2) = \langle x, y, z | x^2y^3z^{-1}yz = 1\rangle$, as a pro-2-group [14, p.359]. The number $|\mathcal{K}_G|/m_G$ can then be computed as the number of quotients of $\mathbf{G}(2)$ isomorphic to G. We computed this quantity using GAP [5] as a check on our field-theoretic computations, finding agreement.

The six octic Galois groups which are embeddable in $\tilde{S}_4$ are $T1 = C_8$, $T4 = D_4$, $T5 = Q$, $T8 = \tilde{D}_4$, $T12 = \tilde{A}_4$, and $T23 = \tilde{S}_4$. From Table 4.1, these account for $24 + 18 + 6 + 36 + 0 + 16 = 100$ octic 2-adic fields. In terms of Galois extensions, $\tilde{S}_4$ contributes only 8. The remaining $130 - 92 = 38$ studied in [1] and [16] come from non-transitive subgroups of S_8, of orders 2, 3, 4, 6, and 12.

4.4. **Connections with number fields.** The last block of columns in the case of Type 24C groups summarizes some connections with number fields. The "2", "$2, p_3$", and "$2, p_5$" columns give the number of octic number fields with discriminant $\pm 2^a$, $\pm 2^a p_3^b$, and $\pm 2^a p_5^b$ respectively, and the indicated Galois group. Here p_3 is any prime congruent to 3 modulo 8 while p_5 is any prime congruent to 5 modulo 8. The independence of the enumeration on the odd prime is part of Example 11.18 of [11], which also says that for all these fields the global Galois group agrees with the 2-decomposition group. The Galois theory of Section 3 applies in these global contexts as well, and so any integer in the m column divides all the integers to its right. The totals of the "2", "$2, p_3$" and "$2, p_5$" columns are respectively 36, 579, and 621, while the total number of Type 24C 2-adic fields is 1499.

Using the results of this paper, it is proved in [19] that the the 579 octic 2-adic fields which globalize to fields of discriminant $\pm 2^a 3^b$ are exactly the same as the 579 fields which globalize in the case $p_3 = 11, 19, \ldots$. Similarly, the 621 octic 2-adic fields which globalize to fields of discriminant $\pm 2^a 5^b$ are exactly the same as the 579 fields which globalize in the case $p_5 = 13, 29, \ldots$. This statement is not contained in Example 11.18 of [11] as indeed this example extends to arbitrary degree 2^k while the stronger statement on 2-adic ramification fails for $k \geq 4$.

Also using the results of this paper, especially the slope information in the next section, it is proved in [7] that the only number fields of degree ≤ 15 unramified outside of two are the octic fields enumerated in Table 4.1 and their subfields. Similarly, [10] analyzes 2-adic ramification of number fields in this degree range but with primes outside of two allowed to ramify.

5. Galois slope content, inertia groups, and root numbers

Our website also gives invariants of fields beyond those that are needed to obtain Table 4.1. §5.1-5.3, §5.4, and §5.5 discuss Galois slope content, inertia groups, and local root numbers respectively.

5.1. **Slopes.** Our approach to measuring higher ramification by slopes is described in detail in [8]. In our approach, only upper numbering of ramification subgroups enters. In the approach in [1] and [16], lower numbering appears naturally. In this subsection, we briefly review our approach.

It is convenient to indicate the degree of a p-adic field by a subscript and the discriminant exponent by a superscript. As explained in [8], a given p-adic field K contains a canonical chain of subfields, starting at $\mathbf{Q}_p = K_1^0$ and ending at $K = K_n^c$. If $K_{n_i}^{c_i} \subset K_{n_{i+1}}^{c_{i+1}}$ are two adjacent members of this chain then the corresponding slope is

$$s = \frac{c_{i+1} - c_i}{n_{i+1} - n_i}. \tag{5.1}$$

Slopes strictly increase as one goes up the canonical chain, and the canonical chain is characterized by being the chain where slopes increase as slowly as possible.

The rational numbers which can arise as slopes are 0, 1, and rational numbers greater than one, corresponding to no ramification, tame ramification, and wild ramification respectively. Given a slope s, let $K_{n_i}^{c_i} \subset K_{n_{i+1}}^{c_{i+1}}$ be the corresponding step in the chain, of relative degree m_s. If s is a wild slope, then m_s must have the form p^{ℓ_s} for ℓ_s a positive integer. We indicate all the slopes by writing

$$\mathrm{SC}(K) = [s_1, \ldots, s_j]_{m_1}^{m_0}, \tag{5.2}$$

with each wild slope s repeated ℓ_s times between the square brackets. Here the superscript is omitted if 0 is not a slope and the subscript is omitted if 1 is not a slope. Also SC stands for slope content.

The theory can be applied not only to the given p-adic field K but also to its splitting field K^g. We put $\mathrm{GSC}(K) = \mathrm{SC}(K^g)$ and talk about the Galois slope content of K. The Galois group $\mathrm{Gal}(K^g/\mathbf{Q}_p)$ has a descending filtration by higher ramification subgroups $\mathrm{Gal}(K^g/\mathbf{Q}_p)^s$ and m_s is the size of the quotient of $\mathrm{Gal}(K^g/\mathbf{Q}_p)^s$ by the next smallest ramification subgroup.

5.2. **Sample computations of Galois slopes.** The website gives $\mathrm{GSC}(K)$ for all the octic 2-adic fields K. In [8] we present several Galois slope computations, including one in the context of octic 2-adic fields. Here we present two more complicated Galois slope computations, so as to more completely illustrate the general method.

As our starting fields we take K_{8a}^{27} and K_{8b}^{27}, defined by

$$f_{8a}(x) = x^8 - 4x^6 - 18x^4 - 16x^2 - 2\,,$$
$$f_{8b}(x) = x^8 - 42x^4 - 72x^2 - 18\,.$$

From Section 3, we know that in each case the Galois group is $T35$, of order $128 = 2^7$. So there are seven slopes to be found in each case, counting a slope s having $m_s = 2^{\ell_s}$ as appearing with multiplicity ℓ_s.

For $* = a, b$ the complete list of subfields is

$$K_1^0 \subset K_{2*}^2 \subset K_{4*}^9 \subset K_{8*}^{27}.$$

So, using (5.1) three times, one has $\mathrm{SC}(K_{8*}^{27}) = [2, 3.5, 4.5]$ for both $* = a$ and $* = b$.

Next we apply our usual octic resolvent from §2.3 to get fields L_{8a}^{24} and L_{8b}^{26} defined respectively by

$$g_{8a}(x) = x^8 + 8x^6 + 72x^4 + 480x^2 + 1296\,,$$
$$g_{8b}(x) = x^8 - 4x^6 + 6x^4 + 12x^2 + 9\,.$$

From Table 3.2 we know that the Galois group is $T29$ in each case. The complete lists of subfields are

$$L_1^0 \subset L_{2a}^3 \subset L_{4a}^{10} \subset L_{8a}^{24}\,,$$
$$L_1^0 \subset L_{2b}^3 \subset L_{4b}^{10} \subset L_{8b}^{26}\,.$$

So, applying (5.1) several times again, one has $\mathrm{SC}(L_{8a}^{24}) = [3, 3.5, 3.5]$ and $\mathrm{SC}(L_{8b}^{26}) = [3, 3.5, 4]$. So far, we have seen five slopes in each case, 2, 3, 3.5, 3.5 and 4.5 for K_{8a}^g and 2, 3, 3.5, 4 and 4.5 for K_{8b}^g.

To find the remaining two slopes we apply our usual octic resolvent again, and get algebras M_{8a} and M_{8b} defined by

$$h_{8a}(x) = \left(x^4 - 6x^2 + 3\right)\left(x^4 - 3x^2 + 3\right)\,,$$
$$h_{8b}(x) = \left(x^4 - 6x^2 + 3\right)\left(x^4 - 6x^2 + 12\right)\,.$$

All four of the printed quartics have Galois group D_4, in conformity with Table 3.2. In the first case, the Galois slope contents of the two quartics are $[2, 3, 3.5]$ and $[2, 2]^2$ respectively. So, using the second quartic alone, we conclude that

$$\mathrm{GSC}(K_{8a}) = [2, \mathbf{2}, 3, \mathbf{3.5}, 3.5, \mathbf{4.5}]^2.$$

Here, we have put three wild slopes in bold, to indicate that they are the three slopes in the original set $\mathrm{SC}(K_{8a})$.

In the second case, the Galois slope contents of the two quartics are $[2, 3, 3.5]$ and $[2, 3]^2$, which is not enough to reach a conclusion. We take the compositum of the quadratic subfield $\mathbf{Q}_2(\sqrt{-2})$ of the first quartic with the second quartic field $\mathbf{Q}_2[x]/(x^4-6x^2+12)$ to get $x^8+2x^6+3x^4-4x^2+1$ with Galois group $T9 = D_4 \times C_2$. From Table 3.2, this field has three quartic subfields, one of which is defined by $x^4 + 2x^3 + 3x^2 - 4x + 1$. This quartic has Galois slope content $[2, 2]^2$. So now we have seen that the slope 2 indeed occurs with multiplicity two, and so

$$\mathrm{GSC}(K_{8b}) = [2, \mathbf{2}, 3, \mathbf{3.5}, 4, \mathbf{4.5}]^2.$$

5.3. **Summarizing slope tables.** Tables 5.1 and 5.2 give the lists of Galois slopes which can arise from octic fields with a given non-primitive Galois group G and a given discriminant exponent c. If 2^a exactly divides $|G|$ then a slopes are listed, counting multiplicities. The slopes which correspond to subquotients of order 3 are either 0 or 1 and explained in the text. Following the convention introduced in the last subsection, the three slopes of K itself are put in boldface while the remaining slopes of K^g are put in ordinary type. So if the three slopes in bold are $s_1 \leq s_2 \leq s_3$ one always has $c = s_1 + 2s_2 + 4s_3$. If two fields K_1 and K_2 are siblings, then the corresponding lists of Galois slopes agree, although the font may be different in the case $|G| = 2^a$.

Except for the type 24P groups $T13$, $T14$, and $T24$, the left block of Table 5.2 consists of type 4 groups and the right block consists of type 2 groups. Each type 4 field is part of a packet of eight, via twisting. A packet is always put on a single line, even when twisting sometimes changes one of the slopes. The octic resolvent of a type 4 field depends only on the packet it is in and is given to the immediate right. For type 4 fields, always the largest slope is lost. In the case of type 24P, a twist packet consists of seven fields and an algebra which is a product of two quartic fields. Either the seven fields all have Galois group $T13$ or six have Galois

TABLE 5.1. Slopes in octic 2-adic fields with $|G| = 2^a$

G	c	Slopes	#
1	0	**0**, **0**, **0**	1
	8	**0**, **0**, **2**	1
	12	**0**, **0**, **3**	2
	22	**0**, **3**, **4**	4
	31	**3**, **4**, **5**	16
2	8	**0**, **0**, **2**	1
	12	**0**, **0**, **3**	2
	16	**0**, **2**, **3**	3
	22	**0**, **3**, **4**	4
	24	**2**, **3**, **4**	8
3	16	**0**, **2**, **3**	1
4	12	**0**, **2**, **2**	2
	16	**0**, **2**, **3**	2
	22	**0**, **3**, **4**	2
	22	**2**, **3**, **3.5**	8
	24	**2**, **3**, **4**	4
5	22	**0**, **3**, **4**	2
	24	**2**, **3**, **4**	4
6	22	**0**, 2, **3**, **4**	2
	24	0, **2**, **3**, **4**	2
	27	**2**, 3, **3.5**, **4.5**	8
	28	2, **3**, **3.5**, **4.5**	8
	31	0, **3**, **4**, **5**	4
	31	2, **3**, **4**, **5**	8
7	8	**0**, **0**, 2, **2**	1
	12	**0**, **0**, 2, **3**	1
	16	**0**, **2**, 3, **3**	4
	22	0, **0**, **3**, **4**	2
	22	**0**, 2, **3**, **4**	4
	31	0, **3**, **4**, **5**	8
	31	2, **3**, **4**, **5**	16
8	16	0, **2**, **2**, **2.5**	2
	18	0, **2**, **2**, **3**	2
	22	**0**, 2, **3**, **4**	2
	24	0, **2**, **3**, **4**	2
	27	**2**, 3, **3.5**, **4.5**	8
	28	2, **3**, **3.5**, **4.5**	8
	31	0, **3**, **4**, **5**	4
	31	2, **3**, **4**, **5**	8
9	16	**0**, 2, **2**, **3**	4
	18	0, **2**, **2**, **3**	4
	18	**0**, **2**, 3, **3.5**	4
	20	**0**, 2, **3**, **3.5**	4
	22	0, **2**, **3**, **3.5**	8
	22	**0**, 2, **3**, **4**	4
	24	0, **2**, **3**, **4**	8
10	8	**0**, **0**, 2, **2**	1
	12	0, **0**, **2**, **2**	1
	12	**0**, **0**, 2, **3**	1
	16	0, **0**, **2**, **3**	1
	16	**0**, 2, **2**, **3**	4
	26	2, **3**, **3.5**, **4**	16
11	16	**0**, **2**, 3, **3**	4
	18	**0**, **2**, 3, **3.5**	4
	20	0, **2**, **3**, **3**	8
	20	**0**, 2, **3**, **3.5**	4
	22	0, **2**, **3**, **3.5**	4
	22	**0**, 2, **3**, **4**	8
	24	0, **2**, **3**, **4**	16
15	16	**0**, 2, **2**, 3, **3**	8
	20	0, **2**, **2**, 3, **3.5**	4
	22	0, **2**, **2**, 3, **4**	4
	24	0, 2, **2**, **3**, **4**	4
	27	0, **2**, 3, **3.5**, **4.5**	16
	28	0, 2, **3**, **3.5**, **4.5**	16
	31	0, 2, **3**, **4**, **5**	24
16	8	**0**, **0**, 2, 2, **2**	1
	12	0, **0**, 2, **2**, **2**	1
	12	**0**, **0**, 2, 2, **3**	1
	16	0, **0**, 2, **2**, **3**	1
	22	0, **2**, 2, **3**, **4**	4
	31	2, **3**, 3.5, **4**, **5**	16
17	16	0, 0, **2**, **2**, **2.5**	4
	18	0, 0, **2**, **2**, **3**	4
	20	**0**, 2, 2, **3**, **3.5**	8
	24	0, 0, **2**, **3**, **4**	8
	24	0, 2, **2**, **3**, **4**	8
	27	**2**, 3, **3.5**, 4, **4.5**	32
	31	2, **3**, 3.5, **4**, **5**	32
18	20	0, **2**, **2**, 3, **3.5**	8
	22	0, 2, **2**, **3**, **3.5**	8
	26	0, 2, **3**, **3.5**, **4**	16
19	14	0, 0, **2**, **2**, **2**	2
	18	0, 0, **2**, **2**, **3**	2
	20	0, 0, **2**, **3**, **3**	4
	20	0, **2**, **2**, 3, **3.5**	4
	22	0, 2, **2**, **3**, **3.5**	4
	26	**2**, 3, **3.5**, 4, **4.25**	16
	28	2, **3**, 3.5, **4**, **4.25**	16
20	8	**0**, **0**, 2, 2, **2**	1
	12	**0**, **0**, 2, 2, **3**	1
	16	0, 0, **2**, **3**, **3**	2
	20	**0**, 2, 2, **3**, **3.5**	4
	28	2, **3**, 3.5, **4**, **4.25**	16
21	12	0, **0**, 2, **2**, **2**	1
	16	0, **0**, 2, **2**, **3**	1
	16	0, **0**, **2**, 3, **3**	2
	18	**0**, 2, **2**, 3, **3.5**	4
	25	**2**, **3**, 3.5, 4, **4.25**	16
26	18	**0**, 2, **2**, 3, 3.5, **3.5**	8
	20	**0**, 2, 2, **3**, 3.5, **3.5**	8
	27	0, **2**, 3, **3.5**, 4, **4.5**	16
	28	0, 2, 2, **3**, **3.5**, **4.5**	32
	31	0, 2, **3**, 3.5, **4**, **5**	32
27	8	**0**, **0**, 2, 2, 2, **2**	4
	12	**0**, **0**, 2, 2, 2, **3**	4
	18	0, **0**, **2**, 3, 3, **3.5**	8
	20	**0**, 2, 2, **3**, 3.5, **3.5**	8
	22	**0**, 2, 2, **3**, 3.5, **4**	8
	30	2, **3**, 3.5, **4**, 4.25, **4.75**	32
	31	2, **3**, 3.5, **4**, 4.25, **5**	32
28	12	0, **0**, 2, 2, **2**, **2**	4
	16	0, **0**, 2, 2, **2**, **3**	4
	18	**0**, 2, **2**, 3, 3.5, **3.5**	8
	20	0, **0**, 2, 3, **3**, **3.5**	8
	22	**0**, 2, 2, **3**, 3.5, **4**	8
	29	2, **3**, **3.5**, 4, 4.25, **4.75**	32
	31	2, **3**, 3.5, **4**, 4.25, **5**	32
29	18	**0**, 2, **2**, 3, 3.5, **3.5**	8
	22	**0**, 2, 2, **3**, 3.5, **4**	8
	24	0, 2, 2, **3**, **3.5**, **3.5**	16
	26	0, 2, 2, **3**, **3.5**, **4**	16
	26	0, **2**, 3, **3.5**, 4, **4.25**	16
	28	0, 2, **3**, 3.5, **4**, **4.25**	32
30	16	0, **0**, 2, **2**, 3, **3**	8
	18	**0**, 2, **2**, 3, 3.5, **3.5**	8
	20	**0**, 2, 2, **3**, 3.5, **3.5**	8
	22	0, **0**, 2, 3, **3**, **4**	8
	29	2, **3**, **3.5**, 4, 4.25, **4.75**	32
	30	2, **3**, 3.5, **4**, 4.25, **4.75**	32
31	20	**0**, 2, 2, **3**, 3.5, **3.5**	8
	22	**0**, 2, 2, **3**, 3.5, **4**	8
	25	0, **2**, **3**, 3.5, 4, **4.25**	16
35	21	0, **2**, **2**, 3, 3.5, 3.5, **3.75**	32
	24	0, 2, **2**, 3, **3.5**, 3.5, **3.75**	32
	25	0, 2, **2**, **3**, 3.5, 4, **4.25**	32
	26	0, 2, **2**, 3, **3.5**, 4, **4.25**	32
	27	0, 2, **2**, 3, 3.5, **3.5**, **4.5**	32
	27	0, 2, **2**, 3, **3.5**, 4, **4.5**	32
	29	0, 2, **3**, **3.5**, 4, 4.25, **4.75**	64
	30	0, 2, **3**, 3.5, **4**, 4.25, **4.75**	64
	31	0, 2, **3**, 3.5, **4**, 4.25, **5**	64

group $T24$ and one has Galois group $T14$. Fields in a twist packet with the same Galois group are again put on one line.

The groups $T13$, $T33$, and $T38$ have an A_3 quotient but not an S_3 quotient. The groups $T14$, $T23$, $T24$, $T34$, $T40$, $T41$, and $T44$ have an S_3 quotient but not an A_3 quotient. The group $T42$, the only one on Table 5.2 with 3^2 exactly dividing its order, has both an A_3 quotient and an S_3 quotient. Corresponding to the 3^b factor of $|G|$, one gets a slope of 0 from an A_3 quotient and a slope of 1 from an S_3 quotient. Note that the groups with just A_3 as a quotient also have A_4 as a quotient and thus a corresponding field has at least two 2's among its wild slopes. Similarly, most groups with just S_3 also have a unique S_4 quotient and thus a corresponding field has at least two $c/3$'s among its wild slopes, with $c = 4$ or $c = 8$. The groups $T40$ and $T33$ have three S_4's as quotients, which account for the contribution $[4/3, 4/3, 8/3, 8/3]_3^2$. The remaining group $T42$ does not have A_4

TABLE 5.2. Slopes in octic 2-adic fields with $|G| = 2^a 3^b$ corresponding to subquotients of size 2.

G	c	Slopes	#
13	(12, 14, 18)	**2**, **2**, (**0**, **2**, **3**)	(1, 2, 4)
23	(10, 12, 16)	0, **1.33**, **1.33**, (**1.5**, **2**, **3**)	(2, 2, 4)
	22	0, **2.67**, **2.67**, **3.5**	8
24	(12, 16)	0, **1.33**, **1.33**, (**2**, **3**)	(2, 4)
	(18, 20)	**0**, **2.67**, **2.67**, (**2**, **3**)	(4, 8)
38	(16, 18)	2, 2, **2**, **2**, (**2.5**, **3**)	(4, 4)
	20	2, **2**, **2**, 3, 3, **3.5**	16
	21	**2**, **2**, 3, 3.5, 3.5, **3.75**,	32
40	22	0, 1.33, 1.33, **2.67**, **2.67**, **3.5**	8
44	(14, 16)	0, **1.33**, **1.33**, 2, 2.33, 2.33, (**2.5**, **3**)	(8, 8)
	17	0, **1.33**, **1.33**, 3, 3.17, 3.17, **3.25**	32
	24	0, 2, **2.67**, **2.67**, 3.33, 3.33, **4**	32
	25	0, **2.67**, **2.67**, 3, 3.83, 3.83, **4.25**	64

G	c	Slopes	#
14	8	**0**, **1.33**, **1.33**	1
	16	**0**, **2.67**, **2.67**	2
33	12	**0**, 2, 2, **2**, **2**	2
	20	2, 2, **2**, **3**, **3**	4
	24	2, 2, **3**, **3.5**, **3.5**	8
34	16	**0**, 1.33, 1.33, **2.67**, **2.67**	1
41	16	0, 1.33, 1.33, **2**, **2.33**, **2.33**	2
	18	0, 1.33, 1.33, **2**, **2.67**, **2.67**	2
	22	0, 1.33, 1.33, **3**, **3.17**, **3.17**	8
	22	0, **2**, 2.67, 2.67, **3.33**, **3.33**	8
	26	0, 2.67, 2.67, **3**, **3.83**, **3.83**	16
42	8	**0**, 1.33, 1.33, **1.33**, **1.33**	1
	16	**0**, 2.67, 2.67, **2.67**, **2.67**	4

as a quotient; nor does it have S_4 as a quotient, despite the fact that wild slopes 4/3 and 8/3 appear.

Finally, the two fields with $G = T25$ have Galois slope content $[2, 2, 2]^7$ while the fourteen fields with $G = T36$ have Galois slope content $[c/7, c/7, c/7]_7^3$. The set of latter fields naturally has the structure of two projective planes over $\mathbf{F}_2$. One plane consists of the fields with c a square modulo seven, thus $c \in \{8, 16, 18\}$. The other consists of the fields with c a non-square, thus $c \in \{10, 12, 20\}$.

The sample computations of the previous subsection began from two of the 64 fields with $(G, c) = (T35, 27)$. From Table 5.1, we see that K_{8a} represents how 32 fields behave while K_{8b} represent how the other the 32 fields behave. All together, Table 4.1 shows that 134 different (e, f, G, c) actually arise with G a 2-group. Table 5.1 says that 126 correspond to a single possibility for Galois slope content while 8 correspond to two possibilities. Similarly, Table 4.1 shows that 33 different (e, f, G, c) arise with $|G| = 2^a 3^b$. Table 5.2 says that 32 give just a single associated Galois slope content, with only $(e, f, G, c) = (8, 1, T41, 22)$ yielding two possibilities.

5.4. **Inertia groups.** The Galois slope content GSC(K) completely describes the subquotients of the slope filtration on $\mathrm{Gal}(K^g/\mathbf{Q}_2)$. For some applications, it is important to go further and have a complete description of the slope filtration itself. Such a description is given in [16] when $\mathrm{Gal}(K^g/\mathbf{Q}_2)$ is embeddable in $\tilde{S}_4$.

Our website takes the first step in going beyond GSC(K) by describing the inertia group $\mathrm{Gal}(K^g/\mathbf{Q}_2)^1$. When K is totally ramified, $\mathrm{Gal}(K^g/\mathbf{Q}_2)^1$ is a transitive group, and the database gives its T-number. When K is not totally ramified, the database gives its isomorphism type.

As a simple example, consider the 384 octic 2-adic fields with Galois group the largest possible 2-group, namely $T35$. Since $T35$ has $T3 = C_2^3$ as a quotient, it must have 0, 2, and 3 as slopes. Since $T35$ does not have C_4 as a quotient, the slope 0 must occur with multiplicity one. Both of these observations are confirmed by Table 5.1 and together they say that $\mathrm{Gal}(K^g/\mathbf{Q}_2)^1$ has index two and thus order sixty-four.

From either Table 4.1 or Table 5.1 one sees that all 384 fields in question are totally ramified, and thus $\mathrm{Gal}(K^g/\mathbf{Q}_2)^1$ is transitive and not $D_4 \times D_4$. There are six possibilities, $T26$ through $T31$. The database shows that they occur with equal frequency as inertia groups, sixty-four times each.

5.5. **Root numbers.** For each octic 2-adic field K, the database gives the root number $\epsilon(K) \in \{1, i, -1, -i\}$ as explained in [8]. These root numbers both assist in classifying fields by means of their invariants and in describing obstructions to embedding problems. The approach in [1] uses root numbers fundamentally. In contrast, we find our fields by other means, as explained in [8]. We then simply compute the root numbers at the end, so as to increase the utility of our database in applications.

As a simple example, consider the fourteen fields with Galois group $T33$, which come in seven twin pairs. Each twin pair consists of one field with root number -1 and one field with root number 1. The field with root number -1 arises as the octic resolvent of $T38$ fields while the field with root number 1 does not.

As a closely related but more complicated example, consider the thirty-six fields K with Galois group $T41$, which come in eighteen twin pairs. These fields are classified by triples (d, K_4, ϵ). Here $\mathbf{Q}_2(\sqrt{d})$ is the quadratic subfield of K, K_4 is the unique S_4-quartic with $K_4^g \subset K^g$, and ϵ is the root number of K. The possibilities for d are the six ramified discriminants, -1, $-*$, 2, $2*$, -2, $-2*$, with $*$ denoting the unramified discriminant. The possibilities for K_4 have defining equations x^4+2x+2, x^4+4x+2, and x^4+4x^2+4x+2 and root numbers $\epsilon_4 = 1$, 1, and -1 respectively. The possibilities for ϵ are 1 and -1. If a $T41$ field K has invariants (d, K_4, ϵ) then its twin has invariants $(*d, K_4, \epsilon_4\epsilon)$. The field K arises as the octic resolvent of $T44$ fields if and only if $(d, *)\epsilon_4\epsilon_8 = 1$, where $(\cdot, \cdot)$ indicates the Hilbert symbol.

The examples just presented and others were originally simply observed from our database. However the behavior of root numbers in these examples also follows from obstruction formulas valid over arbitrary ground fields of characteristic different from two, as proved in [18].

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